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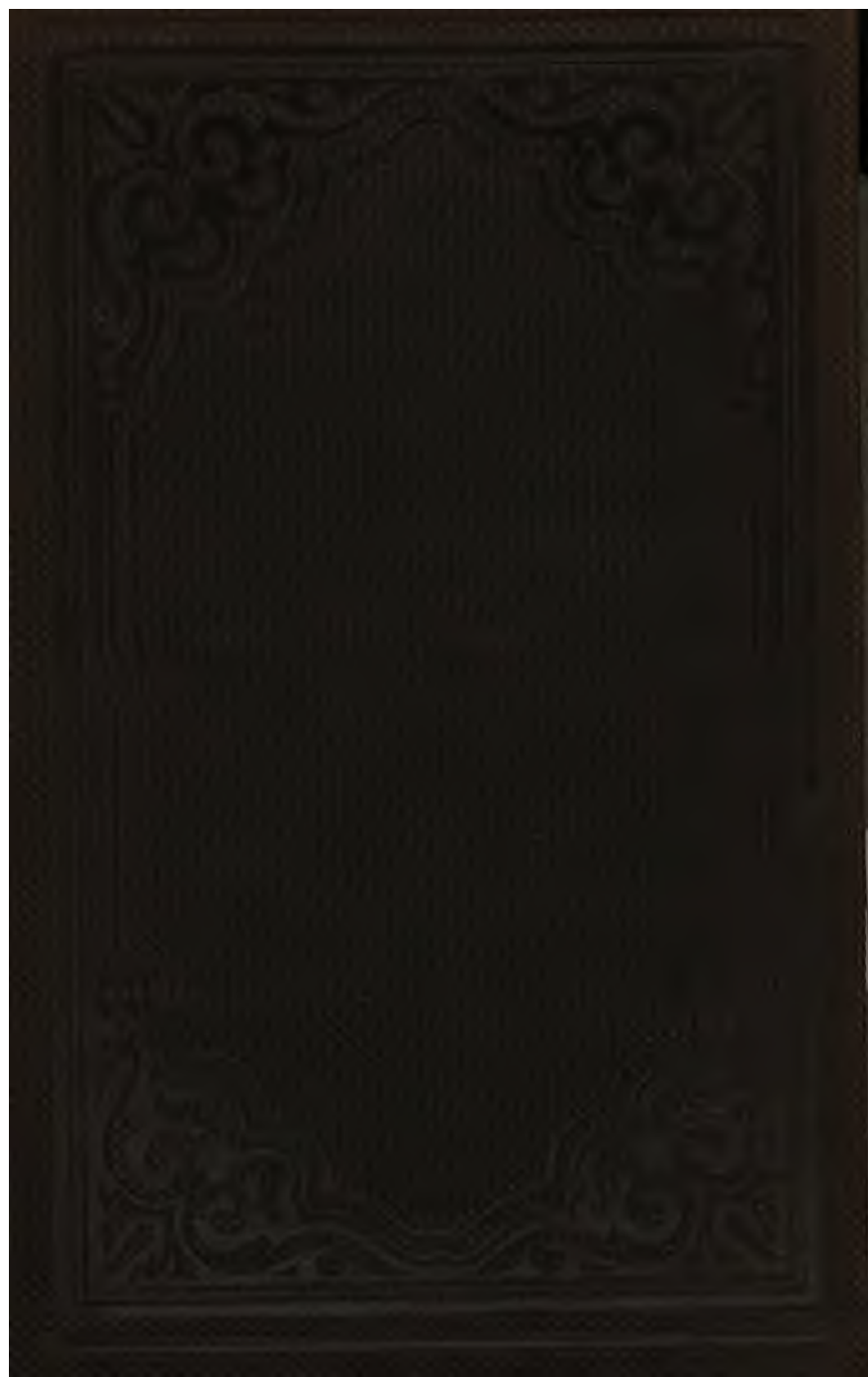
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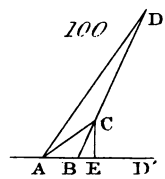
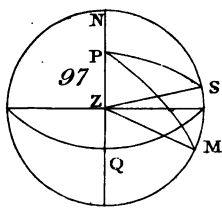
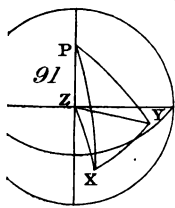
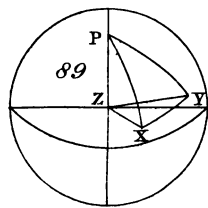
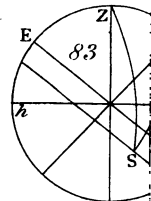
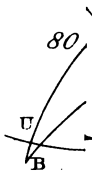
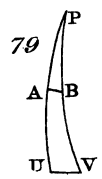
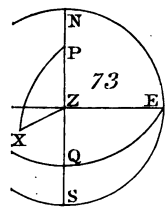
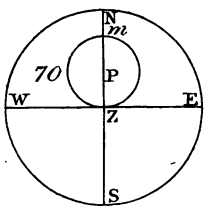
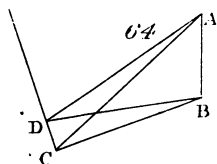
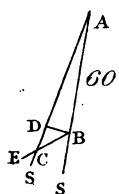
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PLANE AND SPHERICAL TRIGONOMETRY.

PART I.

CONTAINING RULES, EXAMPLES, AND PROBLEMS.

BY

H. W. JEANS, F.R.A.S.

ROYAL NAVAL COLLEGE, PORTSMOUTH;

FORMERLY MATHEMATICAL MASTER IN THE ROYAL MILITARY ACADEMY,
WOOLWICH; AND AN EXAMINER OF MERCHANT OFFICERS IN
NAUTICAL ASTRONOMY, ETC.

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PREFACE.

THE following pages contain the principal Rules in Plane and Spherical Trigonometry. The investigations of these Rules, requiring some knowledge of Mathematics, are given in the Second Part of this treatise. The present volume is made to consist entirely of the Rules and their applications, and the demonstrations are placed in a separate volume, in order that the student may be enabled to proceed with Problems in Surveying, Navigation, and Astronomy, as soon as he is acquainted with vulgar and decimal fractions, and has acquired a sufficient knowledge of algebra to work an easy equation.

The Problems at the end of the book have been selected chiefly for Naval Students; most of them can be solved by means of the Rules contained in the book. The Problems marked with the letter (*a*) have been added for the use of those who have already made some progress in Mathematics: these cannot be solved by a simple application of the Rules; they will present, however, little difficulty to the student who is acquainted with the contents of Part II.

The table of *log. haversines*,* now becoming generally known

* This table, under the name of *logarithmic versed sines*, may be found in Mendoza Rios, calculated to *five* places of decimals; the table in Norie, called *log. rising*, may be formed from it by the addition of the constant $\log. 5.30103$. Dr. Inman re-calculated the above table of Mendoza Rios, and carried it to *six* places of decimals, and arranged it

to the Naval Student, reduces considerably the labour of working out some of the Problems in Navigation: it may also be applied with equal advantage to the principal cases in Plane and Spherical Trigonometry. Rules have, accordingly, been now for the first time adapted to this table. Other rules are also given for the same cases suited to the Logarithmic Tables in more general use, such as Hutton's, or those in Norie's or Riddle's works on Navigation.

The young student who may use this volume as an INTRODUCTION TO NAVIGATION will not find it necessary to read, at first, more than a certain portion of it. The Articles marked in the table of contents with the letter (*n*) may be sufficient for this purpose. All the Examples under each of the Rules thus selected should be worked out, with the exception of a few of the more difficult. Such student may also solve a few of the Problems at the end of the book, such as [1] to [20]; [39] to [47]; and, as an INTRODUCTION TO NAUTICAL ASTRONOMY, Problems [102] to [113]; [135], [136]; [143], [144]. The remainder of the book may be omitted until he has read Part II.

The answers to the Examples and Problems have generally been taken from the tables *by inspection*; that is, if Inman's tables were at hand, to the nearest 15"; if any others, to the nearest minute or half minute. It is seldom necessary in practical questions of this kind to proportion to the nearest second; and in

in a much more convenient form for use; he has inserted it in the last edition of his tables, under the name of *log. haversines*. Lastly, in the work on Navigation by Lieut. Raper, the student will find a similar table, called LOG. SIN² SQUARE. The reason for adopting the two last-mentioned names will appear from considering the formula, by means of which this table may be constructed from the common table of log. sines, namely:

$$\sin.^2 \frac{A}{2} = \frac{1}{2} \text{ ver. } A \text{ [Part II. formula (29)].}$$

$$\text{or, log. sin. square } \frac{A}{2} = \text{log. half versine } A.$$

Raper derived his name from the first side of this equation; Inman from the last, contracting *half versine* into *haversine*.

Nautical Problems the above degree of accuracy will in almost every case be sufficient.

To those who have not the assistance of a tutor, and who have therefore to rely on their own exertions for the knowledge they may require, the Solutions of the Problems, which are published separately, under the title of "Problems in Astronomy, Surveying, and Navigation, with their Solutions," will, it is hoped, be found of service.

H. W. J.

ROYAL NAVAL COLLEGE,
April 27, 1858.

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PLANE TRIGONOMETRY.

CHAPTER I.

(1). ONE of the principal uses of Trigonometry is to furnish rules for finding

Correction.

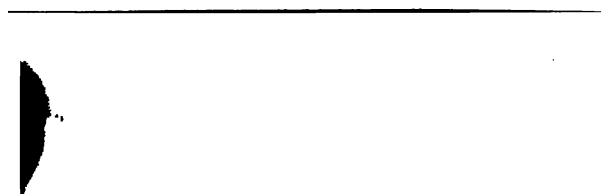
In Prob. 1, p. 74, the secant is taken out instead of the tangent: the answer should be $CB = 89.5$.

usually defined to be, with reference to any triangle, the relations which the sides and angles of triangles have to one another, and of the general relations of angles.

(3). The First Part of this treatise contains the Rules for solving Plane and Spherical Triangles, with a large collection of examples and problems for practice. The problems have been selected principally to serve as an Introduction to Navigation and Nautical Astronomy.

(4). In the Second Part will be investigated the fundamental formulæ in Plane and Spherical Trigonometry. These formulæ continually recur in mathematics, and the student whose views extend beyond the practical part of Navigation must make himself well acquainted with them. In the Second Part also, the rules given in the First Part will be investigated and proved.

(5). At present it will be sufficient to name only a few of



PLANE TRIGONOMETRY.

CHAPTER I.

(1). ONE of the principal uses of Trigonometry is to furnish rules for finding any part of a triangle (that is, an angle or a side), when the number of parts already known is sufficient; which is the case, in general, when three of the parts of the triangle are known.

(2). Considered as a branch of mathematics, Trigonometry teaches us the method of investigating certain relations of angles denominated *formulae*. Trigonometry is therefore usually defined to be, that branch of mathematics which treats of the relations which the sides and angles of triangles have to one another, and of the general relations of angles.

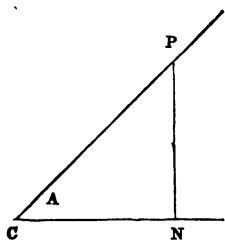
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(5). At present it will be sufficient to name only a few of

the terms used in Trigonometry : the principal of which are the fractions called the *Trigonometrical ratios*.

Definitions of the Trigonometrical ratios.



Let the angle PCN be denoted by A ; and let PN be drawn from any point P perpendicular to CN, thus forming a right-angled triangle : then the trigonometrical ratios are the fractions that can be formed of the three sides of the right-angled triangle PCN, taking them two and two together. It is evident that only six such fractions can be formed in this manner, namely, $\frac{PN}{CP}$, $\frac{OP}{PN}$, $\frac{PN}{CN}$, $\frac{CN}{CP}$, $\frac{CP}{PN}$, and $\frac{CN}{CP}$.* To connect these fractions with one of the angles of the triangle, as A, let us call the side opposite the right angle the hypotenuse, the side opposite to the angle in question (in this case PN) the perpendicular, and the remaining side CN, adjacent to the angle, the base ; then the trigonometrical ratios of an angle are named as follows : the perpendicular divided by the hypotenuse is called the *SINE* of the angle ; the hypotenuse divided by the

* The use and importance of the trigonometrical ratios will be more clearly seen in Part II. We may here briefly remark, that since we are not able to compare together the angles and sides of a triangle (for we cannot say that an angle is three or more times greater than a side, since these quantities have different units of measurement), it has been found convenient to take certain lines or fractions, such as $\frac{PN}{CP}$, &c., and compute their numerical values for different angles, and record these values in tables. Now these fractions will evidently depend for their value on the magnitude of the angles themselves ; and if we know the fraction, we shall know, by means of the table, the angle corresponding to it. Moreover, we can compare the side of a triangle with this fraction, since both quantities are expressed in the same kind of measure, namely, linear measure ; and then, by reference to the table, discover the relation of the side to the angle itself.

perpendicular is called the COSECANT of the angle ; the perpendicular divided by the base is called the TANGENT of the angle ; the base divided by the perpendicular is called the COTANGENT of the angle ; the hypotenuse divided by the base is called the SECANT of the angle ; and the base divided by the hypotenuse is called the COSINE of the angle. Another term used in Trigonometry is the VERSINE of an angle : this quantity is equivalent to the excess of unity over the cosine. These definitions will be more readily learned by writing them down in the form of fractions, thus :

$\frac{PN}{CP}$ or $\frac{\text{perpendicular}}{\text{hypotenuse}}$ is called the *sine* of the angle A (a)

$\frac{CP}{PN}$ or $\frac{\text{hyp.}}{\text{perp.}}$. . . cosecant . . . (b)

$\frac{PN}{CN}$ or $\frac{\text{perp.}}{\text{base}}$. . . tangent . . . (c)

$\frac{CN}{PN}$ or $\frac{\text{base}}{\text{perp.}}$. . . cotangent . . . (d)

$\frac{CP}{CN}$ or $\frac{\text{hyp.}}{\text{base}}$. . . secant . . . (e)

$\frac{CN}{CP}$ or $\frac{\text{base}}{\text{hyp.}}$. . . cosine . . . (f)

Also, $1 - \text{cosine of } A$ is called the *versine* of angle A.

The values of these ratios or fractions being calculated for every minute or quarter of a minute from 0° to 45° , constitute the tables of sines, tangents, &c., by means of which, and certain other tables of numbers called *logarithms*, problems in Trigonometry and Astronomy are usually solved.

(6). Before, therefore, we proceed to apply Rules for solving trigonometrical problems, we will, in order to familiarise the student with logarithmic computation, show their extensive use in shortening and simplifying complex and tedious arithmetical calculations ; in fact, we shall find that, by their means, processes of multiplication are reduced to addition,

division to subtraction, the formation of powers to multiplication, and the extraction of roots to division, &c.

(7). By methods given in Part II., it is shown how the logarithms of numbers have been computed and arranged in tables, so that now the logarithm of any given number may be found by inspection; and conversely, if any logarithm is given, the number corresponding to it (which, for the sake of distinction, is called the natural number) may also be taken out.

LOGARITHMS.

EXAMPLES AND EXERCISES IN THE USE OF LOGARITHMS.

Logarithms usually consist of a whole number and a decimal fraction: thus, the logarithm of the natural number 200 is 2.301030 (see Tables): the whole number, 2, is called the *index* or characteristic; the decimal part, namely $.301030$, is called the *mantissa*. The mantissa is always positive; the index may be positive or negative. Thus, the logarithm 2.301030 is $+2$ and $+.301030$; but the logarithm $\bar{2}.301030$ is -2 and $+.301030$. In the tables the decimal part only is inserted; the index being found as follows:

RULE I.

(8). *To find the index of the logarithm of a whole or mixed number.*

The index of the logarithm is less by one than the number of *integral* places contained in the number.

Thus the <i>index</i> of the log. of	24	is		1
	of 2463	is		3
	of 147.2	is		2
	of 14.72	is		1
	of 1.472	is		0

RULE II.

(9). *To find the index of the logarithm of a decimal fraction.*

Consider the decimal fraction as a whole number, and find

the index of its logarithm by the preceding rule : then subtract the number thus found from the number of decimals in the given fraction ; the remainder, with the negative sign (—) placed over it, is the index required.

Thus, the index of the logarithm of $\cdot 00452$ is $\bar{3}$; for the index of the logarithm of 452 is 2 , by Rule I., and the number of decimals in $\cdot 00452$ is 5 : hence $2-5=-3$, or as it must be written $\bar{3}$, to show that the mantissa or decimal part of the logarithm is still positive. Hence, to reduce a logarithm with a negative index to a quantity wholly negative, we must proceed thus : $\bar{3}\cdot 602060$, or its equivalent $-3+\cdot 602060=-2\cdot 397940$; and therefore $-4\cdot 301415=\bar{5}\cdot 698585$.

EXAMPLES.

The index of log. $\cdot 123$	is	$\bar{1}$
of log. $\cdot 0123$	is	$\bar{2}$
of log. $\cdot 00064$	is	$\bar{4}$
of log. $\cdot 000000721$	is	$\bar{7}$

(10). *To find the index of the logarithm of a vulgar fraction.*

Reduce the fraction to an equivalent decimal fraction : the index of its logarithm may then be found by Rule II.

Thus, the index of log. $\frac{1}{8}$ or of log. $\cdot 125$	is	$\bar{1}$
of log. $24\frac{2}{3}$ or of log. $24\cdot 4$	is	1
of log. $\frac{1}{25}$ or of log. $\cdot 04$	is	$\bar{2}$
of log. $\frac{1}{120}$ or of log. $\cdot 00833$	is	$\bar{3}$

RULE III.

(11). *To take out the logarithm of any given number from the table.*

The rule for taking out a logarithm for any given number will be found in the explanations which accompany the table.

to which the student is referred.* We will here insert a collection of numbers whose logarithms it is required to find from the tables.

EXAMPLES.

Let it be required to take out from the tables the logarithms of 2482, of 24·82, and of ·002482.

Looking for 2482 in the table, we see opposite to it ·394802, the decimal part of its logarithm; and this part, by the explanation accompanying the table, is the same for each of the above numbers: the index or whole number to be prefixed will indicate the number of integral figures in the natural number.

Hence the log. of 2482	is		3·394802
log. of 24·82	is		1·394802
log. of ·002482	is		$\bar{3}$ ·394802

Find from the tables the logarithms of the following natural numbers:

Nat. No.		
24	<i>Answer.</i>	Its logarithm is 1·380211
248	„	 2·394452
2480	„	 3·394452
1476	„	 3·169086
14·06	„	 1·147985
1·406	„	 0·147985
3847	„	 3·585122
38475†	„	 4·585178

* It would be difficult to give in this place a rule for taking out a logarithm, as it must be made to depend on the peculiar arrangement of the table in the student's hands; nor is it necessary, since ample directions for finding the logarithms of numbers are always appended to the table.

† When the natural number (as in this example) consists of more than four figures (the extent of logarithmic tables in general use), the fifth and all subsequent figures can be found by the rules given with the tables (see explanation of tables).

Nat. No.	Ans.	Its logarithm is	
384.75		2.585178	
384.757	„	2.585186	
4196.534	„	3.622890	
121004.15	„	5.082800	
48 $\frac{3}{4}$ or 48.75	„	1.687975	
49 $\frac{1}{4}$ or 49.25	„	1.692406	
2000	„	3.301030	
200000	„	5.301030	
1000 $\frac{1}{2}$	„	3.000216	
2.4	„	0.380211	
.24	„	$\bar{1}$.380211	
.024	„	$\bar{2}$.380211	
.000035	„	$\bar{5}$.544068	
1415216	„	6.150822	
$\frac{3}{4}$ or .75	„	$\bar{1}$.875061	
$\frac{1}{2}$	„	$\bar{1}$.698970	
$\frac{1}{25}$	„	$\bar{2}$.602060	
$\frac{1}{2000}$	„	$\bar{4}$.698970	
.004	„	$\bar{3}$.602060	
.04	„	$\bar{2}$.602060	
.6945	„	$\bar{1}$.841672	
.75234	„	$\bar{1}$.876414	
90 $\frac{1}{25}$	„	1.954435	
1000 $\frac{1}{125}$	„	3.000003	
387 $\frac{1}{2}$	„	2.588272	
726 $\frac{1}{4}$	„	2.861086	

RULE IV.

(12). *To take out the natural number corresponding to any given logarithm.*

For the method of taking out the several figures that constitute the natural number, see explanation accompanying the tables.

To find where the decimal point in the number taken out must be placed.

First. When the index of the given logarithm is positive.

The number of integral places in the number taken out will be *one* more than the index. Thus, if the index of a logarithm is 2, the natural number contains three integral places; and so on: the rest of the figures in the number taken out are decimals.

EXAMPLES.

Given the logarithm 2·477121, to find its natural number.

Entering the table with the decimal part ·477121, we find the number corresponding to it to be 3, or 30, or 300, or 3000, &c.; but, as the index of the logarithm is 2, the natural number must contain three integral figures. Hence the natural number of 2·477121 is 300.

Required the natural numbers of the following logarithms:

Logarithms.	<i>Ans.</i>	Nat. No. is	
1·380211			24
2·394452	"		248
3·394452	"		2480
6·394452	"		2480000
2·415671	"		260·417
1·415674	"		26·042
2·310101	"		204·221
4·196171	"		15709·83
0·217845	"		1·65137
1·841569	"		69·4335
2·841989	"		695·0064
5·082800	"		121004·19
0·147985	"		1·406
0·394452	"		2·48

(13). *Second.* When the index of the given logarithm is negative, to find its natural number.

The natural number in this case will be a decimal. To find the number of ciphers (if any) to be prefixed to the figures taken from the tables, subtract 1 from the index (without regarding the negative sign); the remainder will denote the number of ciphers to be prefixed to the figures of the number taken from the tables: thus, if the index is $\bar{4}$, prefix three ciphers; if $\bar{2}$, prefix one cipher; if $\bar{1}$, the decimal mark is placed next to the figures taken out; and so on.

EXAMPLES.

Given the logarithm $\bar{4}\cdot394452$, to find its natural number.

Entering the table with the decimal part $\cdot394452$, we see the natural number opposite to it is 248: to this number prefix three ciphers, since the index of the logarithm is $\bar{4}$; hence the natural number of $\bar{4}\cdot394452$ is $\cdot000248$.

Required the natural numbers of the following logarithms:

Logarithms.			
$\bar{2}\cdot380211$	<i>Ans.</i>	Nat. No. is	$\cdot024$
$\bar{5}\cdot544068$	„		$\cdot000035$
$\bar{1}\cdot841672$	„		$\cdot6945$
$\bar{7}\cdot875061$	„		$\cdot00000075$
$\bar{3}\cdot602060$	„		$\cdot004$
$2\cdot394452$	„		$\cdot0248$

RULE V.

Multiplication by logarithms.

(14). Take out the logarithms of the given numbers from the tables; add them together: their sum will be the logarithm of the product of the given numbers; the natural number corresponding to which is therefore the product required.

EXAMPLES.

Required the product of the following quantities :

- | | | | |
|------|---------------------------------|-------------|-----------|
| [1]. | $47 \times 1.405 \times 84$ | <i>Ans.</i> | 5546.94 |
| [2]. | $.47 \times 140.5 \times .0084$ | „ | $.5547$ |

Calculation.

[1].		[2].	
log. 47	. . 1.672098	log. .47	. . $\bar{1}.672098$
„ 1.405	. . 0.147676	„ 140.5	. . 2.147676
„ 84	. . 1.924279	„ .0084	. . $\bar{3}.924279$
log. product	. 3.744053	log. product	. $\bar{1}.744053$
∴ product	5546.94	∴ product	$.5547$

Required the product of the following quantities :

- | | | | |
|----|--|-------------|---------------|
| 1. | $72 \times 96 \times 124 \times .05$ | <i>Ans.</i> | 42854 |
| 2. | 84×96 | „ | 8064 |
| 3. | $6 \times 4 \times 12 \times 32$ | „ | 9216 |
| 4. | $64 \times 362 \times .4$ | „ | 9267 |
| 5. | $36 \times 48 \times 62 \times .4$ | „ | 42854 |
| 6. | $1234 \times 9671 \times .00617$ | „ | 73632 |
| 7. | $2.4 \times .007 \times .54 \times .1$ | „ | $.0009072$ |
| 8. | $784 \times .000079 \times .0000036$ | „ | $.0000002229$ |

RULE VI.

Division by logarithms.

(15). From the logarithm of the dividend subtract the logarithm of the divisor : the remainder will be the logarithm of the quotient ; the natural number corresponding to which will be the quotient required.

EXAMPLES.

- | | | | | |
|------|------------|-----------------|-------------|----------|
| [1]. | Divide 472 | by 32.2 | <i>Ans.</i> | 14.66 |
| [2]. | . . | $.0472$ by 3.22 | „ | $.01466$ |

Calculation.

[1].	[2].
log. 472 . . . 2.673942	log. .0472 . . . $\bar{2}$.673942
log. 32.2 . . . 1.507856	log. 3.22 . . . 0.507856
log. quotient . . 1.166086	log. quotient . . $\bar{2}$.166086
$\therefore$ quotient . . 14.66	$\therefore$ quotient . . .01466

Find the value of the following expressions :

9. $68 \div 34$	<i>Ans.</i> 2
10. $96 \div 1.6$	" 60
11. $2004.64 \div 34$	" 58.96
12. $19 \div 72$	" 0.26389
13. $19 \div .72$	" 26.389
14. $\frac{242 \times 559 \times 63}{781 \times 432}$	" 25.26
15. $\frac{84 \times .00769 \times .683}{598 \times .0000146 \times .039}$	" 1295.71
16. $1 \div 3.2$ or $\frac{1}{3.2}$	" .3125
17. $1 \div .45$ or $\frac{1}{.45}$	" 2.222
18. $1 \div .0004572$ or $\frac{1}{.0004572}$	" 2187.23

RULE VII.

Involution, or raising of powers by logarithms.

(16). *First.* When the quantity to be raised to a power is a whole or mixed number.

Multiply the logarithm of the quantity to be raised by the number denoting the power; and the product will be the logarithm of the power, the natural number of which is the power required.

EXAMPLES.

Required the 16th power of 1.05, or the value of $(1.05)^{16}$.

Calculation.

$$\log. 1.05 \quad \dots 0.021189$$

16

 127134

 21189

$$\log. (1.05)^{16} \quad \dots .339024$$

$$\therefore (1.05)^{16} = 2.18285$$

19.	Required the 6th power of 4.7215	<i>Ans.</i> 11078
20.	" " 3d " 12½	" 1953.127
21.	" " 150th " 1.05	" 1507.82
22.	" " 200th " 1.0125	" 11.989
23.	" " 4th " 7½	" 3701.53
24.	" " 1000th " 1.0125	" 247742.3

(17). *Second.* When the quantity to be raised to a power is a decimal fraction.

Take from the tables the logarithm of the decimal fraction : multiply, separately, the decimal part of the logarithm so taken out, and the index of the logarithm, by the number denoting the power.

From the latter product subtract the whole number in the former, placing the negative sign over the remainder ; and then affix to it the decimal part of the former product, the natural number of which will be the required power : which find from the tables.

EXAMPLES.

Required the 10th power of .2, or the value of $(.2)^{10}$.

$$\log. .2 = \bar{1}.301030 = \bar{1} + .301030$$

$$\begin{array}{r} 10 \quad \quad 10 \\ \hline \end{array}$$

$$\begin{array}{r} 10 \quad 3.010300 \\ \hline \end{array}$$

$$\begin{array}{r} 10 \\ \hline \end{array}$$

$$\log. (.2)^{10} \quad \bar{7}.010300$$

$$\therefore (.2)^{10} = .000001024$$

Required the values of the following expressions :

25. The 5th power of .2 or $(.2)^5$	<i>Ans.</i> .00032
26. $(.8)^5$	„ .512
27. $(.09163)^4$	„ .000070494
28. $(.975)^{200}$	„ .0063241

RULE VIII.

Evolution, or extracting of roots by logarithms.

(18). *First.* When the index of the logarithm of the given number is positive ; or, if negative, divisible by the number denoting the root.

Take out the logarithm of the given number, and divide it by the figure denoting the root to be extracted ; and the result will be the logarithm of the root, which find from the tables.

EXAMPLES.

[1]. Required the cube root of 1234.

[2]. Required the 5th root of .00005214.

Calculation.

[1].	[2].
log. 1234 .. 3)3.091315	log. .00005214 .. 5)5.717171
log. $\sqrt[3]{1234} .. 1.030438$	log. $\sqrt[5]{.00005214} .. \bar{1}.143434$
$\therefore \sqrt[3]{1234} .. 10.73$	$\therefore \sqrt[5]{.00005214} .. .1391$

29. Required the 5th root of 784	<i>Ans.</i> 3.79195
30. . . . square root of 365	„ 19.10498
31. . . . cube root of 12345	„ 23.11162
32. . . . 10th root of 2	„ 1.071776
33. . . . square root of .093	„ .304959
34. . . . 5th root of 7.0825	„ 1.479235
35. . . . 365th root of 1.045	„ 1.000121
36. . . . cube root of .00125	„ .1077

(19). *Second.* When the index of the logarithm of the

number is *negative*, and is not divisible by the figure denoting the root.

Take out the logarithm of the number whose root is required; increase the index by the least number of units that will render it divisible by the figure denoting the root to be extracted, annexing to the decimal part the same number of units so added to the index: then proceed as before. Thus, to divide $\bar{4} \cdot 681241$ by 3, write it down thus, $\bar{6} + 2 \cdot 681241$; then $\bar{6} + 2 \cdot 681241$ divided by 3 is $\bar{2} \cdot 893747$, the natural number of which is $\cdot 078297$.

EXAMPLES.

[1]. Required the square root of $\cdot 1452$, or the value of $\sqrt{\cdot 1452}$.

[2]. Required the 10th root of $\cdot 00345$, or the value of $\sqrt[10]{\cdot 00345}$.

Calculation.

[1].	[2].
log. $\cdot 1452 \dots \bar{1} \cdot 161967$	log. $\dots \cdot 00345 \dots \bar{3} \cdot 537819$
or $\bar{2} + 1 \cdot 161967$	or $\bar{10} + 7 \cdot 537819$
log. $\sqrt{\cdot 1452} = \bar{1} \cdot 580983$	log. $\sqrt[10]{\cdot 00345} = \bar{1} \cdot 753782$
$\therefore \sqrt{\cdot 1452} = \cdot 381$	$\therefore \sqrt[10]{\cdot 00345} = \cdot 5672$

37. Required the cube root of $\cdot 0125$	<i>Ans.</i> $\cdot 2321$
38. . . . square root of $\cdot 0093$	„ $\cdot 09644$
39. . . . 72d root of $\cdot 096$	„ $\cdot 96797$
40. . . . cube root of $\cdot 000048$	„ $\cdot 036342$

RULE IX.

To find the value of a number raised to a power represented by a fraction.

(20). Multiply the logarithm of the number by the numerator of the fraction, and divide the result by the denominator: the natural number corresponding to the quotient will be the value required.

EXAMPLES.

- [1]. Required the value of $(6.025)^{\frac{4}{5}}$.
 [2]. $(.0925)^{1.6}$, or $(.0925)^{\frac{8}{5}}$.

Calculation.

[1].	[2].
log. 6.025 . . 0.779957	log. .0925 . . $\bar{2}.966142$
<u>4</u>	<u>3</u>
7)3.119828	2)4.898426
log. $(6.025)^{\frac{4}{5}} = 0.445689$	log. $(.0925)^{\frac{8}{5}} = \bar{2}.449213$
$\therefore (6.025)^{\frac{4}{5}} = 2.79054$	$\therefore (.0925)^{\frac{8}{5}} = .02813$

Required the value of the following expressions :

- | | |
|--|----------------------|
| 41. $(.096)^{\frac{2}{3}}$ | <i>Ans.</i> 0.272016 |
| 42. $(19)^{\frac{2}{3}}$ | „ 10.5439 |
| 43. $\frac{(466871)^{\frac{2}{3}} \times \sqrt[3]{(3576)^{16}}}{996003 \times \sqrt{.0077}}$ | „ 1717739 |
| 44. The cube root of $(.001234)^2$ or $(.001234)^{\frac{2}{3}}$ | „ .01159 |
| 45. $(472)^{\frac{2}{3}}$ | „ 30.586 |
| 46. $(.042)^{.83}$ | .000000000003741 |
| 47. $(.00563)^{.07}$ | .6958825 |

(21). Rules 7, 8, and 9 are derived from the same property of logarithms, namely, $\log. a^n = n \log. a$, where n may be any number, either whole or fractional: thus $\log. 6^5 = 5 \log. 6$; $\log. \sqrt[3]{7} = \frac{1}{3} \log. 7$; $\log. 2^{\frac{3}{4}} = \frac{3}{4} \log. 2$.

(22). If the index n of the quantity be negative, it will be perhaps the easiest way to reduce it, in the first place, to an equivalent expression with a positive index, which may be always done by substituting for the given quantity its reciprocal, with the sign of the power changed. Thus it is proved in Algebra that

$$a^{-n} = \frac{1}{a^n}, \quad x^{-3} = \frac{1}{x^3}, \quad 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}}, \quad \frac{1}{2^{-5}} = 2^5, \quad \frac{a^{-n}}{b^{-p}} = \frac{b^p}{a^n}: \text{ hence the}$$

$\log. \text{ of } a^{-n} = \log. \frac{1}{a^n} = \log. 1 - \log. a^n = -n \log. a$ [since
 $\log. 1 = 0$ (Part II.)]; $\log. x^{-3} = \log. \frac{1}{x^3} = -3 \log. x$;
 $\log. 4^{-\frac{1}{3}} = \log. \frac{1}{4^{\frac{1}{3}}} = -\frac{1}{3} \log. 4$; $\log. \frac{1}{x^{-5}} = \log. x^5 = 5 \log. x$;
 $\log. \frac{a^{-n}}{b^{-p}} = \log. \frac{b^p}{a^n} = p \log. b - n \log. a$; and so on.

EXAMPLES.

Required the value of the following expressions :

$$[1]. 4^{-6} \qquad [2]. \frac{1}{4^{-5}} \qquad [3]. \frac{1}{.2^{-4}}$$

Calculation.

$$\begin{array}{lll}
 4^{-6} = \frac{1}{4^6} & \frac{1}{4^{-5}} = 4^5 & \frac{1}{.2^{-4}} = .2^4 \\
 \log. 1 \dots 0.000000 & \log. 4 \dots 0.602060 & \log. 2 \dots \bar{1}.301030 \\
 \log. 4^6 \dots \bar{2}.010300 & & \phantom{\bar{1}.301030} 5 4 \\
 \log. 4^{-6} \dots 1.989700 & \log. \frac{1}{4^{-5}} \dots 3.010300 & \phantom{\bar{1}.301030} 3.204120 \\
 \therefore 4^{-6} = 97.656 & & \text{or } \bar{7} + 4.204120 \\
 & \therefore \frac{1}{4^{-5}} = 1024 & \therefore \log. .2^4 \dots \bar{1}.600589 \\
 & & \therefore .2^4 = .3987
 \end{array}$$

Examples to the preceding rules.

Required the value of the following expressions :

48.	3^7	Ans.	2187
49.	$3^{\frac{1}{2}}$	"	1.16993
50.	3^{-7}	"	.0004572
51.	$3^{-\frac{1}{2}}$	"	.5648
52.	$3^{-\frac{1}{4}}$	"	.5774
53.	$(4.2)^{\frac{1}{2}}$	"	1.507
54.	$(.045)^{-\frac{1}{2}}$	"	1.859
55.	$(.045)^{-\frac{1}{3}}$	"	2.425

56.	$\frac{1}{.2^{-4}}$	Ans.	·0016
57.	$\frac{1}{(.02)^{-\frac{1}{2}}}$	"	·04373
58.	$\frac{3^7}{9^2 \times 4^{-6}}$	"	110591·3
59.	$\frac{3^7 \times 4^{-9}}{3^{-2}}$	"	·07509
60.	$\frac{4^{-5} \times 5^{-\frac{3}{2}}}{3^5 \times \sqrt{2}}$	"	·00000675
61.	$\frac{7\sqrt{15}}{.015} \times .0139 \sqrt{\frac{2}{.11}}$	"	107·124
62.	$\frac{33 \times 45 \times 14}{3^5 \times \sqrt{2}}$	"	420·022
63.	$3^{2\frac{1}{2}}$ and $(4^2)^3$	"	15·5884 and 4096
64.	$(4)^{2^3}$ and $(4)^{2^{\frac{1}{2}}}$	"	65536 and 7·103
65.	$(50)^{.5^{10}}$	"	1·00383
66.	$(6)^{\sqrt{5}}$	"	54·96
67.	$(3124)^{\log .2}$	"	11·3

(23). Sometimes examples of the following kind occur, in which several of the terms are connected by the signs + or -. In such cases the value of each term must be found separately, and applied according to its sign, as in the following

EXAMPLE.

Required the value of the following fraction :

$$\frac{\{6\sqrt{47\cdot5} + 3\sqrt[3]{147\frac{1}{2}}\}^{\frac{2}{3}} \cdot (48\frac{1}{2})^{\frac{1}{3}}}{3(1\cdot05)^{20} - 100 \times \cdot004}$$

The value of this expression is found as follows:

Calculation of numerator.

log.	47·5..	1·676694	log.	147·5..	2·168792
∴ log.	$\sqrt{47\cdot5}$..	0·838347	∴ log.	$\sqrt[3]{147\cdot5}$..	0·722931
log.	6 ..	0·778151	log.	3 ..	0·477121
∴ log.	$6\sqrt{47\cdot5}$..	1·616498	∴ log.	$3\sqrt[3]{147\frac{1}{2}}$..	1·200052

$$\begin{array}{rcl}
 \therefore 6\sqrt{47.5} = 41.35 & & \therefore 3\sqrt[3]{147\frac{1}{2}} = 15.85 \\
 \text{and } \therefore 6\sqrt{47.5} + 3\sqrt[3]{147.5} = 57.2 & & \\
 \log. 57.2 \dots 1.757396 & & \log. 48.75 \dots 1.687975 \\
 & 3 & 4 \\
 & \hline
 & 5) 5.272188 & 5) 6.751900 \\
 \log. (57.2)^{\frac{3}{2}} \dots 1.054437 & & \log. (48\frac{3}{4})^{\frac{3}{2}} \dots 1.350380 \\
 & & \text{and } \log. (57.2)^{\frac{3}{2}} \dots 1.054437 \\
 & & \therefore \log. \text{ numerator} = 2.404817
 \end{array}$$

Calculation of denominator.

$$\begin{array}{rcl}
 \log. 1.05 \dots 0.021189 & & \\
 & 20 & \\
 \log. (1.05)^{20} \dots 0.423780 & & \\
 \log. 3 \dots 0.477121 & & \\
 \log. 3 (1.05)^{20} \dots 0.900901 & & \log. \text{ numerator} \dots 2.404817 \\
 \therefore 3 (1.05)^{20} = 7.96 & & \log. \text{ denominator} \dots 0.878522 \\
 \text{and } 100 \times .004 = .40 & & \therefore \log. \text{ fraction} \dots 1.526295 \\
 \therefore \text{denominator} = 7.56 & & \text{and fraction} = 33.6 \text{ Ans.}
 \end{array}$$

Required the value of the following expressions :

$$\begin{array}{rcl}
 68. \frac{2(\frac{3}{2})^{15} - 2}{\frac{3}{2} - 1} & & \text{Ans. } 1747.6 \\
 69. \left(\frac{\frac{5}{9} \cdot \sqrt[18]{2}}{\frac{10}{17} \cdot \sqrt[2]{3}} + 5^{\frac{2}{3}} \right)^3 & & \text{,, } 94.794 \\
 70. \frac{\sqrt[5]{\frac{7}{3}} \sqrt[6]{6} + 253 \sqrt[3]{\frac{716.5}{\sqrt{2}}} - 20}{19\frac{1}{2} - 4.54} & & \text{,, } 333.03
 \end{array}$$

RULE X.

Of four proportional quantities, any three being given to find the fourth.

(24). If the required quantity be an extreme term, add together the logarithms of the two middle terms, and subtract the logarithm of the other extreme.

If the required quantity be one of the middle terms, add the logarithms of the extremes, and subtract the logarithm of the other middle term.

Thus, if $a : b :: c : x$ and x be required, add together $\log. b$ and $\log. c$, and from the sum subtract $\log. a$; the remainder will be $\log. x$. If $a : b :: x : c$, add together $\log. a$ and $\log. c$, and from the sum subtract $\log. b$; the remainder is $\log. x$.

The student will find it convenient in working an equation or proportion by logarithms to make a dash or stroke with the pen under the term required.

EXAMPLES.

Given $6 : 12 :: 50 : x$, to find x .

Calculation.

$$\begin{array}{r}
 \log. 12 \dots\dots 1\cdot079181 \\
 \log. 50 \dots\dots 1\cdot698970 \\
 \hline
 2\cdot778151 \\
 \log. 6 \dots\dots 0\cdot778151 \\
 \hline
 \log. x \dots\dots 2\cdot000000 \\
 \therefore x = 100
 \end{array}$$

Find the value of x in the following proportions :

- | | |
|--|-------------------------------|
| 71. $24 : 35 :: 79 : x$ | <i>Ans.</i> $x = 115\cdot208$ |
| 72. $3505 : x :: 1507 : 29\cdot8$ | „ $x = 69\cdot3$ |
| 73. Find a fourth proportional to $\cdot0963$, $\cdot24958$, and $\cdot008967$ | <i>Ans.</i> $\cdot02324$ |
| 74. $\sqrt{724} : \sqrt{\frac{5}{13}} :: 6\cdot927 : x$ | „ $\cdot1596$ |

RULE XI.

To reduce arithmetical and algebraical expressions to logarithms.

- (25). (a) The logarithm of the product of any number of

terms is equal to the sum of the logarithms of all the terms in the product: thus, if $x=ab$, then

$$\log. x = \log. a + \log. b$$

(b) The logarithm of the *quotient* of any two numbers is equal to the logarithm of the dividend diminished by the logarithm of the divisor: thus, if $x=a \div b$, or $\frac{a}{b}$, then

$$\log. x = \log. a - \log. b$$

If $x = \frac{abc}{de}$, then

$$\log. x = \log. a + \log. b + \log. c - \log. d - \log. e$$

(c) The logarithm of the *power* of any quantity is equal to the logarithm of the quantity multiplied by the number denoting the power: thus, if $x=a^{10}$, then

$$\log. x = 10 \log. a$$

If $x=a^2b^3$, then

$$\log. x = 2 \log. a + 3 \log. b$$

(d) The logarithm of the *root* of any quantity is equal to the logarithm of the quantity divided by the number denoting the root to be extracted: thus, if $x=\sqrt[3]{a}$, or $x=a^{\frac{1}{3}}$, then

$$\log. x = \frac{\log. a}{3} \text{ or } \frac{1}{3} \log. a$$

If $x=a^{\frac{2}{3}}b^{\frac{1}{3}}$, then

$$\log. x = \frac{2}{3} \log. a + \frac{1}{3} \log. b$$

Examples of reducing algebraical expressions to logarithms.

Reduce the following expressions to logarithms:

75. $x=abcd$ *Ans.* $\log. x = \log. a + \log. b + \log. c + \log. d$

76. $x = \frac{ab}{c}$ „ $\log. x = \log. a + \log. b - \log. c$

77. $x = \frac{ab}{cd}$ „ $\log. x = \log. a + \log. b - \log. c - \log. d$

78. $x = a^2bcd^2$ „ $\log. x = 2 \log. a + \log. b + \log. c + 2 \log. d$

79. $x = \frac{a^2b\sqrt[3]{c}}{10}$ „ $\log. x = 2 \log. a + \log. b + \frac{1}{3} \log. c - 10$

$$80. \quad x = \frac{\sqrt{a} \cdot b \sqrt[4]{c}}{d} \quad \text{Ans. } \log. x = \frac{1}{2} \log. a + \log. b + \frac{1}{4} \log. c - \log. d$$

$$81. \quad x = \frac{\sqrt{ab}}{cd^2} \quad \text{,, } \log. x = \frac{1}{2} \log. a + \frac{1}{2} \log. b - \log. c - 2 \log. d$$

$$82. \quad x = \frac{\sqrt[3]{abc^4}}{\sqrt[4]{d}} \quad \text{,, } \log. x = \frac{1}{3} \log. a + \frac{1}{3} \log. b + \frac{4}{3} \log. c - \frac{1}{4} \log. d$$

(26). By means of the preceding rules the following equations may be solved.

Find an approximate value of x in the following equations:

$$83. \quad 10^x = 456 \quad \text{Ans. } x = 2.658965$$

$$84. \quad x^3 = 14 \quad \text{,, } x = 2.4101$$

$$85. \quad x^5 = 14.76 \quad \text{,, } x = 1.71323$$

$$86. \quad 3^{2x} = 20 \quad \text{,, } x = 1.3634$$

$$87. \quad x^3 = .004 \quad \text{,, } x = .1587$$

$$88. \quad x^{-3} = 4\frac{1}{2} \quad \text{,, } x = .6057$$

$$89. \quad x = (.02445)^{\frac{2}{3}} \quad \text{,, } x = .06183$$

$$90. \quad x = \sqrt[2000]{47691} \quad \text{,, } x = 1.0054$$

$$91. \quad x = \sqrt{\frac{2}{123}} \quad \text{,, } x = .1275$$

$$92. \quad x = \sqrt[3]{\frac{1}{3.14159}} \quad \text{,, } x = .6827$$

$$93. \quad a^x = b \quad \text{,, } x = \frac{\log. b}{\log. a}$$

$$94. \quad \frac{a^x}{b^x} = c \quad \text{,, } x = \frac{\log. c}{\log. a - \log. b}$$

$$95. \quad \frac{a^{mx}}{b^{nx-1}} = c \quad \text{,, } x = \frac{\log. c - \log. b}{m \log. a - n \log. b}$$

$$96. \quad a^x = \frac{b^{mx-n}}{c^{rx}} \quad \text{,, } x = \frac{n \log. b}{m \log. b - \log. a - r \log. c}$$

$$97. \quad \left(\frac{5}{4}\right)^x = 54\frac{1}{2} \quad \text{,, } x = 17.91$$

$$98. \quad a^{\frac{b}{x}} = c \quad \text{,, } x = \frac{b \log. a}{\log. c}$$

$$99. \quad a^{b^x} = c \quad \text{,, } x = \frac{1}{\log. b} \cdot \log. \left(\frac{\log. c}{\log. a} \right)$$

$$\begin{array}{ll}
 100. & (10\frac{1}{2})^x = 20 \qquad \text{Ans. } x = 1.27 \\
 101. & 2^x = 4 \qquad \qquad \qquad \text{,, } x = .6309
 \end{array}$$

RULE XII.

To find the value of a fractional logarithmic expression, when the index of the logarithm of one or both of the terms is negative.

Reduce the logarithm with the negative index to a quantity wholly negative (Rule II. p. 4). The logarithm then becomes wholly negative, and the division can be performed by the common algebraical rule.

EXAMPLE.

Given $(.04)^{3x} = 5$; find the numerical value of x .

Since $(.04)^{3x} = 5$

$$\therefore 3x \log .04 = \log .5$$

$$3x = \frac{\log .5}{\log .04} = \frac{0.698970}{2.602060} = \frac{.69897}{-1.39794} = -0.5$$

$$\therefore x = -1.6666$$

Find the numerical value of x in the following equations :

$$4^{3x} = .005$$

$$\text{Ans. } x = -1.274$$

$$(.04)^{3x} = .001$$

$$\text{,, } x = .715$$

$$(.04)^{x^2} = 5$$

$$\text{,, } x = \sqrt{-(0.5)}$$

$$(.04)^{\sqrt{x}} = 5$$

$$\text{,, } \sqrt{x} = -0.5$$

(27). Logarithms were originally invented to facilitate trigonometrical calculations. The following examples will show their use in applications of another kind.

IN GEOMETRICAL PROGRESSION. If a = the first term of a geometrical series, r = common ratio, n = number of terms, and S = sum of terms; then it is proved in Algebra that $S = a \cdot \frac{r^n - 1}{r - 1}$. Any three of these quantities being given, the fourth may be readily found by means of logarithms.

EXAMPLES.

102. Find the sum of 20 terms of the series,

$$S = a \cdot \frac{r^n - 1}{r - 1} = 1 \cdot \frac{\left(\frac{3}{2}\right)^{20} - 1}{\frac{3}{2} - 1} = \frac{(1.5)^{20} - 1}{\frac{1}{2}}$$

$$\log. (1.5)^{20} = 20 \log. 1.5 = 3.521820$$

$$\therefore (1.5)^{20} = 3325.21$$

$$\text{and } S = \frac{3325.21 - 1}{\frac{1}{2}} = \frac{3324.21}{\frac{1}{2}} = 6648.42. \quad \text{Ans.}$$

103. The sum of a geometrical series is 6560, its first term 2, and common ratio 3: find the number of terms.

$$S = a \frac{r^n - 1}{r - 1} \quad \therefore 6560 = 2 \cdot \frac{3^n - 1}{3 - 1} = 3^n - 1$$

$$\therefore 3^n = 6561, \text{ and } n \log. 3 = \log. 6561, \therefore n = \frac{\log. 6561}{\log. 3} = 8$$

104. If sum of series = 1023, first term = 1, common ratio = 2; what is the number of terms? *Ans.* $n = 10$.

Without the use of logarithms the last two questions would be very difficult to answer.

IN COMPOUND INTEREST. If P denote the principal, r the interest of £1 for 1 year, n the number of years, and A the amount, then it is proved in Algebra that $A = P(1+r)^n$: any three of these quantities being given, the fourth may be found.

EXAMPLES.

105. If £200 be placed out at compound interest for 7 years at 4 per cent: required the amount.

$$A = P(1+r)^n$$

$$\therefore \log. A = \log. P + n \log. (1+r) = \log. 200 + 7 \log. (1.04)$$

$$\text{Ans. } A = £263 \text{ 3s. 8d.}$$

106. At what rate of interest must £400 be placed out that it may amount to £569 6s. 8d. in nine years, at compound interest?

$$A = P(1+r)^n \quad \therefore \log. (1+r) = \frac{\log. A - \log. P}{n}$$

Ans. $r = .04$, or 4 per cent.

107. In how many years will £500 amount to £900 at 5 per cent, compound interest?

$$A = P(1+r)^n \quad \therefore n = \frac{\log. A - \log. P}{\log. (1+r)}$$

Ans. $n = 12.04$ years.

108. IN ASTRONOMY. It is proved that the *squares* of the times which two planets employ to make their revolutions about the Sun are to each other as the *cubes* of their distances from the same heavenly body. Knowing that the revolution of the Earth about the Sun is performed in 365 days, 5 hours, 48 minutes, 51 seconds; and that of Jupiter in 4330 days, 14 hours, 39 minutes, 2 seconds: it is required to find the ratio of the distances of these planets from the Sun.

Reducing in the first place into seconds the given revolutions, we have 31556931 seconds for the Earth, and 374164742 for Jupiter. Representing the first by a , the second by b , the distance of the Earth from the Sun by 1, and that of Jupiter by x , we have

$$x^3 : 1^3 :: b^2 : a^2$$

$$\text{or } x^3 = \frac{b^2}{a^2}$$

$$\therefore 3 \log. x = 2 \log. b - 2 \log. a$$

$$\text{and } \log. x = \frac{2 \log. b - 2 \log. a}{3} = 0.715978$$

$$\therefore x = 5.1997$$

Or the distance of Jupiter from the Sun is to that of the Earth from the same body nearly as 52 : 10.

Table of logarithmic sines, &c. Table of natural versines.

(28). In addition to the table of logarithms of numbers, the use of which has been illustrated by so many examples, the student must make himself acquainted with the table of

log. sines, tangents, &c.; as by means of these two tables nearly the whole of the problems in Trigonometry and Navigation are solved.

Another table, called the table of natural versines, is sometimes used with advantage in questions where an angle or arc is required to the nearest second; since, this table being computed to seconds, the value of the arc or angle may be taken out by inspection, and thus the tedious labour of proportioning for seconds, which it is sometimes necessary to do when the table of log. sines is used, is avoided.

As rules and directions for taking quantities out of these tables to the nearest second accompany the tables, we will give in this place only a few examples as exercises.

Table of log. sines, tangents, &c.

EXAMPLES.

113. Take out the log. sine, log. tangent, and log. cosecant of the following angles (to the nearest second):

Angles.		Log. sine.	Log. tan.	Log. cosec.
10° 10' 6"	<i>Ans.</i>	9·246845	9·253720	10·753155
19 10 40	„	9·516536	9·541332	10·483464
48 35 35	„	9·875079	10·054613	10·124921
61 24 40	„	9·943532	10·263630	10·056468

114. Required the angles (to the nearest second) whose log. sines are:

9·641452	<i>Ans.</i>	25° 58' 30"
9·714185	„	31 11 12
9·984204	„	74 38 26

In the above examples the value of the angle is found to the nearest second; but in the common problems in Navigation it is seldom necessary to proportion to the nearest second, it being sufficiently correct to take out the quantity required *by inspection*. In Inman's Nautical Tables, the log. sine, &c.

of an angle can be taken out by inspection to the nearest 15"; in nearly all other works on Navigation, to the nearest minute.

(29). When the angle whose trigonometrical ratio is required is greater than 90° , its value may be found in the table as follows: Subtract the angle from 180° , and look for the remainder, which is called its *supplement*, in the tables. Thus, to find the log. sine of $100^\circ 10'$, subtract it from 180° , and look for the log. sine of the remainder (namely, $79^\circ 50'$), which is 9.993127; therefore $\log. \sin 100^\circ 10' = 9.993127$.

(30). But the readiest way will, in general, be to *diminish* the given angle by 90° , and to look out the remainder according to the following rule:

RULE XIII.

To take out from the table the log. sine, log. cosine, &c. of an angle greater than 90° .

Diminish the angle by 90° : then

if sine is required look out cosine of remainder

cosine sine

tangent cotangent.....

and in the same manner for secant, cosecant, &c.

EXAMPLES.

Take out of the tables the log. cosine of 100° and the log. cosecant of $170^\circ 14' 15''$.

$$\log. \cos. 100^\circ = \log. \sin. 10^\circ$$

$$\log. \text{cosecant } 170^\circ 14' 15'' = \log. \sec. 80^\circ 14' 15''$$

By the tables:

$$\log. \sin. 10^\circ = 9.239670 \text{ and}$$

$$\log. \secant 80^\circ 14' 15'' = 10.770665$$

$$\therefore \log. \cos. 100^\circ = 9.239670 \text{ and}$$

$$\log. \text{cosecant } 170^\circ 14' 15'' = 10.770665.$$

(31). *Table of natural versines.*

EXAMPLES.

115. Find the natural versines of the following angles:

26° 32' 15"	Ans.	105357
157 48 50	"	1925964
90 7 15	"	1002109
125 0 30	"	1573695

116. Required the angles whose natural versines are :

1175443	Ans.	100° 6' 16"
105357	"	26 32 15
1925964	"	157 48 50
1573695	"	125 0 30

Other special tables used in Trigonometry and Navigation require no examples in this place to illustrate their use.

(32). *Reduction of trigonometrical formulæ to tabular logarithms.*

Trigonometrical formulæ are reduced to logarithms by the common rules for reducing algebraical formulæ (25); but in order to avoid, as much as possible, the inconvenience of using negative indices in calculations, the table of log. sines, co-sines, &c. is constructed by adding 10 to the indices of the log. sines, &c. Thus, the sine 30° is proved in Trig. Part II. to be equal to $\frac{1}{2}$ or $\cdot 5$, its logarithm is therefore $\bar{1}\cdot 698970$: but the log. sine of 30° contained in the tables, or as it is called the *tabular* log. sine, is $9\cdot 698970$; that is, the tabular log. sine is equal to the log. sine + 10, and therefore the log. sine of any angle = tab. log. sine - 10. And as the same addition is made to the indices of the logs. of all the trigonometrical ratios, we must, in reducing such trigonometrical formulæ to tabular logarithms, subtract 10 from the logarithm of each trigonometrical ratio used in the expression.

For a similar reason, viz. to avoid the use of negative indices, the table of natural versines is constructed by multiplying the natural versines by one million; thus, by Part II., versine $60^\circ = \frac{1}{2} = \cdot 5$; but the versine of 60° as contained in the table = $\cdot 5 \times 1000000 = 500000$; that is, tab. vers. = ver. $\times 1000000$; therefore, in logarithms,

$$\begin{aligned}\log. \text{ tab. vers.} &= \log. \text{ vers.} + 6 \\ \text{and } \log. \text{ vers.} &= \log. \text{ tab. vers.} - 6\end{aligned}$$

Hence the following rule :

RULE XIV.

To reduce a logarithmic formula to tabular logarithms.

Subtract 6 from each log. versine, and 10 from the logarithm of each of the other trigonometrical terms that occur in the formula.

EXAMPLES.

117. Thus, if $\tan. A = \sin. A \cdot \sec. A$. In tabular logarithms,
 $\text{tab. log. tan. } A - 10 = \text{tab. log. sin. } A - 10 + \text{tab. log. sec. } A - 10$
 or, as it is usually written (suppressing the word tabular, it being understood that the formula is reduced to *tabular* logarithms),

$\log. \tan. A - 10 = \log. \sin. A - 10 + \log. \sec. A - 10$
 and collecting and cancelling the tens,

$$\log. \tan. A = \log. \sin. A + \log. \sec. A - 10$$

If $\tan. A = \frac{\sin. A}{\cos. A}$. In tab. logs.

$$\begin{aligned}\log. \tan. A - 10 &= \log. \sin. A - 10 - (\log. \cos. A - 10) \\ \text{or } \log. \tan. A &= \log. \sin. A + 10 - \log. \cos. A\end{aligned}$$

118. If $\text{ver. } x = 2 \sin. b \cdot \sin. c \cdot \sin.^2 \frac{A}{2}$. In tab. logs.

$$\log. \text{ver. } x - 6 = 301030 + \log. \sin. b - 10 + \log. \sin. c - 10 + 2 \log. \sin. \frac{A}{2} - 20$$

$$\text{or } \log. \text{vers. } x = 6 \cdot 301030 + \log. \sin. b + \log. \sin. c + 2 \log. \sin. \frac{A}{2} - 40$$

EXAMPLES.

Reduce to tabular logarithms the following formulæ :

$$119. \sin. x = \text{cosec. } y \cdot \tan. z \qquad 120. a = b \cdot \tan. A$$

$$121. \tan. A = \frac{a}{b} \qquad 122. \frac{x}{a} = \cot. A$$

Answers to 119, 120, 121, 122.

$$119. \log. \sin. x = \log. \operatorname{cosec}. y + \log. \tan. z - 10$$

$$120. \log. a = \log. b + \log. \tan. A - 10$$

$$121. \log. \tan. A = 10 + \log. a - \log. b$$

$$122. \log. x = \log. a + \log. \cot. A - 10$$

In almost every problem in Trigonometry and Navigation we shall have occasion to reduce trigonometrical formulæ to tabular logarithms. We will therefore give in this place a few examples for reduction, and for finding the value of the unknown quantity by assigning numerical values to the given quantities.

123. Given $\frac{x}{a} = \cot. A$; find the numerical value of x when $a = 20$, and angle $A = 30^\circ 10'$.

$$\text{Since } \frac{x}{a} = \cot. A$$

$$\therefore x = a \cot. A$$

$$\text{In tab. log. } \dots \log. x = \log. a + \log. \cot. A - 10.$$

Calculation.

$a = 20$	$\log. a \dots\dots 1.301030$
$A = 30^\circ 10'$	$\log. \cot. A \dots 10.235648$
	11.536678
	10
	$\log. x \dots\dots 1.536678$
	$\therefore x = 34.41$

Given $\tan. x = \frac{\sin. A \cos. B}{\tan. C}$; to find x , when $A = 20^\circ 10'$, $B = 32^\circ 45'$, and $C = 78^\circ 45'$.

$$\tan. x = \frac{\sin. A \cos. B}{\tan. C}$$

In tabular logarithms,

$$\begin{aligned} \log. \tan. x - 10 &= \log. \sin. A - 10 + \log. \cos. B - 10 - (\log. \tan. C - 10) \\ &= \log. \sin. A - 10 + \log. \cos. B - 10 - \log. \tan. C + 10 \end{aligned}$$

Cancelling the tens, we have

$$\log. \tan. x = \log. \sin. A + \log. \cos. B - \log. \tan. C$$

Calculation.

A=20° 10'	log. sin. A ..	9.537507
B=32 45	„ cos. B ..	9.924816
C=78 45		19.462323
	„ tan. C ..	10.701338
	„ tan. x ..	8.760985
		∴ x=3° 18'

* In the equation, $x = a \cdot \tan. A \cdot \sin. B \cdot \operatorname{cosec}. C$, find x , having given $a=416$ feet, $A=23^\circ 50' 15''$, $B=54^\circ 28' 30''$, and $C=147^\circ 32' 50''$. *Ans.* $x=278.7$ feet.

(33). *Use of the algebraic signs + and - to determine the magnitude of an angle.*

The numerical value of any trigonometrical ratio of an angle and of its supplement (29) being the same, therefore the log. of any trigonometrical ratio taken out of the tables corresponds to both angles: thus, $\sin. 100^\circ$ being equal to $\sin. 80^\circ$, the $\log. \sin. 100^\circ = \log. \sin. 80^\circ$. For this reason, when we have given the log. sine of an angle to find the angle, we are not always certain that the angle found in the tables or its supplement is the correct one. This uncertainty, however, can be removed when the trigonometrical ratio in question is a tangent, cotangent, secant, or cosine, by finding, by the following Rule, whether its algebraic sign is + or -. When the algebraic sign is +, the angle taken out of the tables is the one sought: when the algebraic sign is -, we must subtract the angle taken out from 180° for the required angle.

RULE XV.

(34). To determine the algebraic sign of a trigonometrical ratio in any formula adapted to logarithms where all the other terms are known, and thence to find the magnitude of the angle, that is, whether it is greater or less than 90° .

Write down the formula, and simplify it by clearing it of fractions, &c.; put over each given term its proper algebraical sign; that is, over each tangent, secant, cosine, and cotangent of an angle or arc less than 90° , the sign +; and over each of the same quantities when the angle is greater than 90° , the sign -; but over each sine, cosecant, and versine, the sign +, whether the angle is greater or less than 90° : and determine from thence the sign of the product of that side whose terms are all known.

Then, since the sign of the product of each side of the equation must be the same (otherwise we should have a positive quantity equal to a negative quantity), make it so by putting over the unknown term the sign + or - accordingly.

Then, if + falls over the unknown term, the part required is less than 90° , and the quantity taken out will be the angle required; but if -, subtract the angle taken out of the table from 180° , the remainder will then be the angle required.

(*Ambiguous case*). When the part sought is expressed in terms of the sine, the above rule will not apply, since the sine is positive, whether the angle is greater or less than 90° . The uncertainty which thence arises (forming what is called in Trigonometry the *ambiguous case*) can be removed only in particular cases.

In the above rule the angle required is supposed to be less than 180° .

EXAMPLES.

In each of the following formulæ it is required to find the value of x : supposing $A=45^\circ$, $B=120^\circ$, and $C=130^\circ$.

123*. Given $\cos. x = \tan. A \cdot \cos. B \cdot \sec. C$, find in the first place whether x is greater or less than 90° .

By the rule we see that $\tan. A$ is +, $\cos. B$ is —, and $\sec. C$ is —: placing, therefore, over the given terms their proper signs, we have

+ — —

$\cos. x = \tan. A . \cos. B . \sec. C$; and since the product of the three terms on the right-hand side of the equation is positive, $\cos. x$ must also be positive; put the sign + over $\cos. x$ accordingly, which shows that x must be less than 90° : hence

+ + — —

the formula becomes $\cos. x = \tan. A . \cos. B . \sec. C$.

Calculation.

$\log. \cos. x - 10 = \log. \tan. A - 10 + \log. \cos. B - 10 + \log. \sec. C - 10$

or $\log. \cos. x = \log. \tan. A + \log. \cos. B + \log. \sec. C - 20$

$A = 45^\circ$	$\log. \tan. A \dots$	10.000000
$B = 120$	„ $\cos. B \dots$	9.698970
$C = 130$	„ $\sec. C \dots$	10.191933
	„ $\cos. x \dots$	9.890903
	$\therefore x = 38^\circ 56'$	

124. Given $\sec. x = \sin. A . \sec. B$, to find whether x is greater or less than 90° .

Determining by the rule the signs of $\sin. A$ and $\sec. B$, we see that the sign — must be placed over $\sec. x$, or

— + —

$\sec. x = \sin. A . \sec. B$, a negative product; therefore $\sec. x$ is negative, or x is greater than 90° .

Calculation.

$\log. \sec. x = \log. \sin. A + \log. \sec. B - 10$.

$A = 45^\circ$	$\log. \sin. A \dots$	9.849485
$B = 120$	„ $\sec. B \dots$	0.301030 (rejecting 10)
	„ $\sec. x \dots$	10.150515
		45°
		180
	$\therefore x =$	135

125. $\tan. x = \csc. A \cdot \cos. B$

Placing over $\csc. A$ and $\cos. B$ their proper algebraic signs, we see that $\tan. x$ is negative, or

$$\tan. x = \csc. A \cdot \cos. B$$

and therefore x is greater than 90° .

126. $\csc. A \cdot \sin. B = \cot. x$

Proceeding in a similar manner, we find that x is less than 90° .

127. $\sec. A \cdot \sin.^2 C = \cos.^2 B \cdot \cot. x$

Placing over the given quantities their proper signs, and re-

collecting that the square of $\cos. B$ must be positive, we have

$$\sec. A \cdot \sin.^2 C = \cos.^2 B \cdot \cot. x$$

. $\therefore x$ is less than 90° .

128. $\sec. A \cdot \sin.^2 C = -\cos.^2 B \cdot \cot. x$

Placing the signs over the given quantities, and taking into consideration the negative sign in front of the right-hand side of the equation, we see that $-$ must be placed over $\cot. x$ in order to render the right-hand side of the equation the same sign as the left, namely, positive : hence

$$\sec. A \cdot \sin.^2 C = -\cos.^2 B \cdot \cot. x$$

. $\therefore x$ is greater than 90° .

129. $\sin. A \cdot \cos. x = -\cos. B \cdot \cot. C$

. $\therefore x$ is greater than 90° .

130. $\text{ver. } A = \frac{\cos. B}{\sin. C \cdot \tan. x}$

or $\text{ver. } A \cdot \sin. C \cdot \tan. x = \cos. B$

or x is greater than 90° .

(35). Sometimes trigonometrical expressions occur in which several of the terms are connected by the signs $+$ or $-$,

and which cannot be reduced to a convenient logarithmic form. To determine the numerical value of such expressions, the value of each term must be found separately, and then applied with its proper sign. The method of proceeding will be seen in the following example:

Given $u = \sqrt[3]{\tan. x \cdot \sec. y - a^2 b \cdot \cos.^2 y}$; to find the value of u , when $x = 32^\circ 10'$, $y = 80^\circ$, $a = 25$, and $b = 50$.

Ans. $u = -9.79$.

Calculation.

Let $m = \tan. x \sec. y$

„ $n = a^2 b \cos.^2 y$ $\therefore u = \sqrt[3]{m-n}$

$\log. m = \log. \tan. x + \log. \sec. y - 20$

$\log. n = 2 \log. a + \log. b + 2 \log. \cos. y - 20$

and $\log. u = \frac{1}{3} \log. (m-n)$

To find m .

$\log. \tan. x \dots\dots 9.798596$

$\log. \sec. y \dots\dots 10.760330$

$\log. m \dots\dots 0.558926$

$\therefore m = 3.62$

To find u .

$\log. u = \frac{1}{3} \log. (m-n)$

$\log. (m-n) \dots\dots 2.972517$

$\therefore \frac{1}{3} \log. (m-n) \dots\dots 0.990839$

$\therefore u = -9.79$

To find n .

$\log. a \dots\dots 1.397940$

$\log. a \dots\dots 1.397940$

$\log. b \dots\dots 1.698970$

$\log. \cos. y \dots\dots 9.239670$

$\log. \cos. y \dots\dots 9.239670$

$\log. n \dots\dots 2.974190$

$\therefore n = 942.3$

and $m = 3.62$

$\therefore m-n = -938.68$

Find u in the following expressions: the values of a , b , x , and y , being the same as in the last example.

$u = a \cos. x - b \sin. y$

Ans. $u = -28.08$

$u = \frac{a \sec. x + b \cos.^2 y}{\tan. x - \cos. y}$

„ $u = 39.6$

The following applications of logarithms to the solution of some important problems in Nautical Astronomy and Navigation may not be unacceptable to the Naval student. The

investigations or proofs of these problems will be found in *Navigation*, Part II. The results only are given in this place, as exercises in the use of logarithms.

(36). To calculate the DIP of the horizon.

The dip may be computed from this formula :

$$\text{dip (in minutes)} = \sqrt{.984a}$$

where a is the height of the eye in feet above the surface of the sea.

EXAMPLE.

Required the dip of the horizon for the following heights of the eye above the sea: (1), 110 feet; (2), 20 feet.

Ans. 10' 22"; 4' 26".

(37). To calculate the PARALLAX of a heavenly body.

The horizontal parallax (P) may be determined by observing the zenith distances of a heavenly body at two distant places on the same meridian; for it can be proved that

$$P \text{ (in seconds)} = \frac{z+z_1-(l+l_1)}{2} \operatorname{cosec.} \frac{z+z_1}{2} \sec. \frac{z-z_1}{2}$$

where z and z_1 are the reduced zenith distances in latitudes l and l_1 .

EXAMPLE.

Required the horizontal parallax of Mars from the following observations :

In lat. l 59° 20' 30" N. $z = 68^\circ 14' 6''$

„ l_1 33 55 0 S. $z_1 = 25 \quad 2 \quad 0$

Ans. $P = 22''.93$.

(38). To calculate the AUGMENTATION of the Moon's semidiameter.

The augmentation (A) of the Moon's horizontal semidiameter for any altitude (t) may be computed from this formula :

$$A \text{ (in sec.)} = 2R \cdot \sec. t \cdot \sin. \frac{1}{2}(t+\alpha) \cdot \sin. \frac{1}{2}(t-\alpha)$$

where R = horizontal semidiameter, t = true altitude, and α = apparent altitude.

EXAMPLE.

The Moon's horizontal semidiameter being $15' 16''$, required its augmentation when the apparent altitude $= 60^\circ 10'$, and true altitude $= 60^\circ 38'$. *Ans.* $A = 13'' \cdot 2$.

(39). To calculate the CONTRACTION of the Moon's semidiameter.

The contraction of the Moon's semidiameter on account of refraction may be computed from this formula :

$$\text{contraction (in '')} = c \cdot \sin.^2 \alpha$$

where c = difference of refraction for the altitudes of vertex and center, and α = inclination of semi. to horizon.

EXAMPLE.

Find the contraction of that semidiameter of the Moon which is inclined 60° ; the difference of refraction for vertex and center being $25''$. *Ans.* Contraction $= 18'' \cdot 75$.

(40). To calculate the REDUCED LATITUDE.

The reduction of the latitude, l , on account of the spheroidal figure of the Earth may be computed from this formula :

$$\text{reduction (in minutes)} = 11 \sin. 2 l$$

EXAMPLE.

The true latitude of a place being 54° , required the reduced latitude. *Ans.* Red. lat. $= 53^\circ 49' 36''$.

(41). To calculate the REDUCED HORIZONTAL PARALLAX.

The Moon's horizontal parallax put down in the Nautical Almanac is the *equatorial* horizontal parallax.

For a place in latitude l , the correction of equatorial horizontal parallax, on account of spheroidal figure of Earth, may be computed from this formula :

$$\text{reduction (in '')} = \cdot 0032 h \cdot \sin.^2 l$$

where h is the Moon's equatorial hor. par.

EXAMPLE.

Required the Moon's horizontal parallax in latitude $50^{\circ} 48' N.$, the hor. par. in Naut. Alm. being $59' 54''$ or $3694''$.

Ans. $59' 47''.1$

(42). To calculate the Earth's EQUATORIAL AND POLAR DIAMETERS.

From the actual measurement of a degree of the meridian in different parts of the world; it appears that the length of the degrees of latitude increases from the equator to the pole, according to a law which would hold if the Earth were an oblate spheroid. The lengths D and D_1 (in feet) of a degree in latitudes l and l_1 being given, the values of the polar and equatorial radii a and b (expressed in miles) can be computed by the following formula:

$$a - b = .0036171 (D - D_1) \operatorname{cosec}. (l + l_1) \operatorname{cosec}. (l - l_1)$$

$$a = .0036171 \{ 2(D - D_1) + 3(D_1 \sin^2 l - D \sin^2 l_1) \} \operatorname{cosec}. (l + l_1) \operatorname{cosec}. (l - l_1)$$

EXAMPLE.

The lengths of a degree of the meridian measured in England and India in latitudes $52^{\circ} 35' 45'' N.$ and $12^{\circ} 32' 21'' N.$ were 364971 feet and 363013 feet respectively; required a and b .

Ans. $a - b = 12.2$ miles; $a = 3962$ miles, $b = 3950$ miles.

The investigation of the formulæ in this important problem may be found in the volume of *Problems in Astronomy, Surveying, and Navigation*, published by the author; which also contains the solutions of all the problems contained in this book. The proofs of the following practical rules are given in *Trigonometry*, Part II.

CHAPTER II.

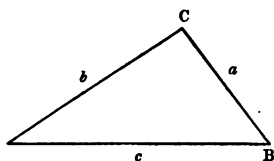
RULES IN PLANE TRIGONOMETRY.

RULE I.—FIRST METHOD, USING HAVERSINES.*

Three sides of a plane triangle being given, to find an angle.

Put down the two sides containing the required angle, and take the difference, under which put the third side; take the sum and difference, and also the half sum and half difference.

To the arithmetical complements† of the logarithms of the first two terms in this form add the logarithms of the last two, and reject 10 in the index; the result will be the log. haversine of the required angle, which find in the table.



EXAMPLE.

In the plane triangle‡ ABC, given the side $a=20$, $b=30$, and $c=40$; required angle A.

* If the student have no table of log. haversines, the angle may be found by the second method.

† The arithmetical complement of a logarithm is *the difference between the logarithm and 10*; thus $10 - \log. 3$ is the arithmetical complement of $\log. 3$. In practice it is most easily found by taking each figure of the logarithm from 9, except the last, and that from 10; thus, ar. co. of 2.714152 is 7.285848; ar. co. 1.314150 is 10.685850 (the last figure in this Example being 5, the cipher at the end not being of any value in that position).

‡ The sides of a triangle are generally distinguished by the same letters as the angles opposite to them, but small, as in the above figure. The diagrams in this, and the following problems, are not drawn to scale: the method of constructing triangles by scale and compass, although not giving such correct results as the logarithmic method used

30	 ar. co. log.	8.522879
40	 ar. co. log.	8.397940
<u>10</u>	log. 15	1.176091
20	log. 5	0.698970
30	log. hav. A	8.795880
10	∴ angle A =	28° 57' 15"
15		
5		

RULE I.—SECOND METHOD, WITHOUT HAVERSINES.

Three sides of a plane triangle being given, to find an angle.

Put down the two sides containing the required angle, and take the difference, under which put the third side; take the sum and difference, and also the half sum and half difference.

To the arithmetical complements of the logarithms of the first two terms in this form, add the logarithms of the last two, divide by 2, and look out the result as a log. sine: the angle corresponding to which will be half the required angle.

EXAMPLES.

[1]. In the triangle ABC, given $a=512$, $b=627$, $c=430$: required C.

512	 ar. co. log.	7.290730
627	 ar. co. log.	7.202732
<u>115</u>	log. 272.5	2.435366
430	log. 157.5	2.197281
<u>545</u>		2)19.126109
315	log. $\sin \frac{1}{2}C$	9.563054
<u>272.5</u>	∴ $\frac{1}{2}C$	21° 26' 45"
157.5		2
	and C =	42 53 30

in this book, ought to be known by the student. He will find ample directions for solving triangles instrumentally in any work on practical geometry and mensuration.

[2]. In the triangle ABC, given $a = .025$, $b = \frac{1}{8}$ or $.125$, and $c = .115$; required B.

$.025$	 ar. co. log.	11.602060
$.115$	 ar. co. log.	10.939302
$.090$	log. $.1075$	$\bar{1}.031408$
$.125$	log. $.0175$	$\bar{2}.243038$
$.212$		$2)19.815808$
$.035$	log. $\sin. \frac{1}{2}B$	9.907904
$.1075$		$\therefore \frac{1}{2}B = 53^\circ 59' 30''$
$.0175$		2
	and B =	$107 \ 59 \ 0$

In the above examples the angles have been taken out of the tables by inspection, namely, to the nearest $15''$. If greater accuracy be required, the student is referred to the rules for proportioning for seconds which accompany his tables of logarithms.

EXAMPLES.

Required the angles of any plane triangle ABC whose three sides, a , b , c , are given, namely:

131.	$a = 798$	<i>Ans.</i>	$A = 89^\circ 45' 37''$
	$b = 460$		$B = 35 \ 12 \ 7$
	$c = 654$		$C = 55 \ 2 \ 16$
132.	$a = 512$	„	$A = 54 \ 8 \ 11$
	$b = 627$		$B = 82 \ 58 \ 11$
	$c = 430$		$C = 42 \ 53 \ 38$
133.	$a = 649$	„	$A = 56 \ 4 \ 6$
	$b = 586$		$B = 48 \ 31 \ 4$
	$c = 757$		$C = 75 \ 24 \ 50$
134.	$a = 627$	„	$A = 29 \ 44 \ 2$
	$b = 1140$		$B = 115 \ 36 \ 32$
	$c = 718.9$		$C = 34 \ 39 \ 26$
135.	$a = .025$	„	$A = 10 \ 58 \ 0$
	$b = \frac{1}{8}$		$B = 107 \ 58 \ 50$
	$c = .115$		$C = 61 \ 3 \ 10$

136.	$a=.8$	<i>Ans.</i>	$A=107^{\circ} 46' 6''$
	$b=.672$		$B= 53 \quad 7 \quad 24$
	$c=.275$		$C= 19 \quad 6 \quad 30$
137.	$a=\frac{1}{2}$	"	$A= 28 \quad 14 \quad 14$
	$b=\frac{3}{4}$		$B= 45 \quad 12 \quad 34$
	$c=1.013$		$C=106 \quad 33 \quad 12$
138.	$a=\frac{1}{4}$	"	$A= 20 \quad 11 \quad 24$
	$b=.541$		$B= 48 \quad 19 \quad 11$
	$c=.674$		$C=111 \quad 29 \quad 23$

In the above examples the angles have been computed to the nearest second; a tedious operation, which the student may omit (the result correct to the nearest minute being, in most cases, considered sufficient).

RULE II.

Of two sides and two opposite angles, any three being given, to find the fourth.

Write down a proportion, having for the first two terms the two sides concerned, and for the third term the sine of the angle opposite to the first side put down, and for the fourth term the sine of the angle opposite the side in the second term of the proportion: mark the term required with a stroke of the pen underneath.

If the term marked be a *middle* term; add the logarithms of the two extreme terms, and subtract the logarithm of the middle term not marked.

If the term marked be an *extreme* term; add the logarithms of the two middle terms, and subtract the logarithm of the extreme term.

The result will be the logarithm of the required term. See examples to Art. (24), p. 19.

NOTE [1]. When two angles of a plane triangle are known, the third can be found by adding together the two known angles and subtracting the result from 180° ; the remainder will be the third angle: hence, if two angles and the adjacent

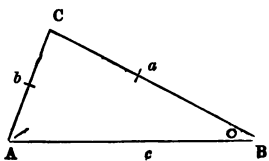
side be given, the third angle is also known, and thence by the above rule the other two sides can be found.

[2]. This rule depends on the property of triangles, that the sides are to one another in the same proportion as the *sines* of the angles opposite to them; and it will be in general the easiest way of working, to write down such proportion, and find the unknown term as in Art. (24).

EXAMPLES.

In a plane triangle ABC, given $a=50\frac{1}{2}$, $b=25$, and $A=68^\circ 48'$: to find the other parts.

Describe the triangle ABC, making the given parts as near the truth as possible without using instruments: mark the given parts (A, a , b) of the triangle with a stroke of the pen, as in the figure: then, since the rule requires that the four parts concerned should be two sides and the two angles opposite to them, the part to be found must be B; mark this with a different character (these marks will render it more easy to apply the rule); then write down a proportion as directed by the rule, and proceed to find B.



The angles A and B being known, C is easily found (Note [1]); and thence, by a proportion, the side c .

Calculation.

[1]. To find B.

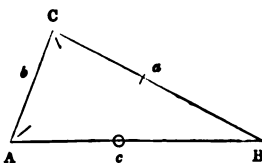
$$\begin{array}{rcl}
 a : b :: \sin. A : \sin. B & & \\
 \log. b \dots 1.397940 & & \\
 \log. \sin. A \ 9.969567 & & \\
 \hline
 & 11.367507 & \\
 \log. a \dots 1.701136 & & \\
 \hline
 \log. \sin. B \ 9.666371 & & \\
 B=27^\circ 38' 15'' & &
 \end{array}$$

[2]. To find C.

$$\begin{array}{rcl}
 B \dots 27^\circ 38' 15'' & & \\
 A \dots 68 \ 48 \ 0 & & \\
 \hline
 & 96 \ 26 \ 15 & \\
 & 180 & \\
 \hline
 C=83^\circ 33' 45'' & &
 \end{array}$$

[3]. To find the side c .

Describe the triangle ABC again, but mark the given parts A , a , and C with a stroke of the pen, and the required part c with a different mark, as in the figure; then make a proportion of the four parts marked; thus



$$\begin{array}{rcl}
 c : a :: \sin. C : \sin. A & & \\
 \log. a \dots\dots\dots & 1.701136 & \\
 \log. \sin. C \dots\dots\dots & 9.997253 & \\
 & \hline & 11.698389 \\
 \log. \sin. A \dots\dots\dots & 9.969567 & \\
 \log. c \dots\dots\dots & 1.728822 & \\
 \hline & \therefore c = 53.56 &
 \end{array}$$

EXAMPLES.

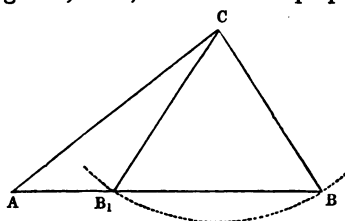
Find the other parts of the triangle ABC , having given :

- | | | | |
|------|--|-------------|---|
| 139. | $a=214$
$b=191$
$A=41^\circ 19' 15''$ | <i>Ans.</i> | $B=36^\circ 6' 30''$
$C=102^\circ 34' 15''$
$c=316.3$ |
| 140. | $a=17.25$
$c=10\frac{3}{4}$ or 10.75
$A=47^\circ 0' 30''$ | „ | $C=27^\circ 7' 15''$
$B=105^\circ 52' 15''$
$b=22.69$ |
| 141. | $a=96$
$c=48$
$A=101^\circ 41'$ | „ | $b=73.98$
$B=49^\circ 0'$
$C=29^\circ 19'$ |
| 142. | $c=376$
$A=48^\circ 3'$
$B=40^\circ 14'$ | „ | $C=91^\circ 43'$
$a=279.7$
$b=243.0$ |
| 143. | $A=60^\circ$
$B=72^\circ$
$a=242$ | „ | $C=48^\circ 0'$
$c=207.6$
$b=265.76$ |
| 144. | $a=2\frac{1}{4}$
$A=43^\circ 24' 10''$
$B=48^\circ 33' 10''$ | „ | $C=88^\circ 2' 40''$
$b=3$
$c=4$ |

145.	$b = \frac{3}{4}$ $a = \frac{1}{2}$ $B = 45^\circ 10'$	<i>Ans.</i>	$A = 28^\circ 13'$ $C = 106^\circ 37'$ $c = 1.013$
146.	$A = 20^\circ 10'$ $B = 48^\circ 15'$ $a = \frac{1}{4}$	„	$C = 111^\circ 35'$ $b = .541$ $c = .6743$
147.	$a = .02$ $A = 11^\circ 32' 15''$ $C = 78^\circ 27' 45''$	„	$B = 90^\circ$ $c = .09797$ $b = .1$
148.	$A = 22^\circ 20' 0''$ $B = 49^\circ 27' 30''$ $c = 2\frac{1}{2}$	„	$C = 108^\circ 12' 30''$ $a = 1$ $b = 2$

The Ambiguous Case.

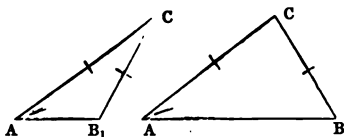
(43). When two sides and the angle opposite the *less* side are given, each of the quantities sought will have two distinct values. For suppose that, in the triangle ABC, the sides BC and AC and the angle A, opposite to the *less* side BC, are given; then, if BC be not perpendicular to AB, from C as a



center, with radius CB, describe an arc cutting AB in another point B_1 , and join CB_1 ; it is manifest that there will be *two* triangles, ACB and ACB_1 , having the sides AC, CB, and angle A,

in one triangle, and the sides AC, CB_1 , and angle A, in the other, of the same magnitude, while the remaining parts are different. Hence it is evident that when such parts of a plane triangle are given, the results will admit of two different values. This may also be shown by considering that the angle ABC or AB_1C is found by the formula $a : b :: \sin. A : \sin. B$, or $a : b :: \sin. A : \sin. AB_1C$; therefore the sines of the angles B and AB_1C have the same numerical value (each being $\frac{b \cdot \sin. A}{a}$): they are consequently the *supplements* of each

other. This also appears from the above figure; for since $CB=CB_1$, the angles B and CB_1B are equal; therefore AB_1C , the supplement of CB_1B , is also the supplement of B . To find these two results we must, therefore, proceed as follows: Having found the acute angle B by the rule, subtract it from 180° for the angle AB_1C in the other triangle; the remaining angles ACB , ACB_1 , and sides AB , AB_1 , may then be found in the usual manner.



EXAMPLE.

In the triangle ABC , or AB_1C , given $a=232$, $b=345$, and $A=37^\circ 20'$; to find the other parts.

Calculation of angle B.

$$\begin{array}{r} a : b :: \sin. A : \sin. B \\ \log. b \dots\dots\dots 2.537819 \\ \log. \sin. A \dots\dots 9.782796 \\ \hline \phantom{} \phantom{} \phantom{} 12.320615 \\ \log. a \dots\dots\dots 2.365488 \\ \log. \sin. B \dots\dots 9.955127 \\ \hline \text{First solution, } B=64^\circ 24' \end{array}$$

There are two solutions.

$$\begin{array}{l} \text{In the first,} \\ B=64^\circ 24' \\ \text{in the second,} \\ B_1=115^\circ 36' \end{array}$$

$$\text{Second solution, } B_1=115^\circ 36'$$

Calculation of angle C.

$$\begin{array}{r} C=180^\circ - A - B \\ A= 37^\circ 20' \\ B= 64 24 \\ \hline 101 44 \\ 180 \\ \hline \therefore C= 78 16 \end{array}$$

Calculation of angle C.

$$\begin{array}{r} C=180^\circ - A - B_1 \\ A= 37^\circ 20' \\ B_1=115 36 \\ \hline 152 56 \\ 180 \\ \hline \therefore C= 27 4 \end{array}$$

Calculation of side c.

$$\begin{array}{rcl}
 c : a :: \sin. C : \sin. A & & \\
 \log. a \dots\dots 2.365488 & & \\
 \log. \sin. C \dots 9.990829 & & \\
 \hline
 & 12.356317 & \\
 \log. \sin. A \dots 9.782796 & & \\
 \hline
 \log. c \dots\dots 2.573521 & & \\
 \therefore c = 374.6 & &
 \end{array}$$

Calculation of side c.

$$\begin{array}{rcl}
 c : a :: \sin. C : \sin. A & & \\
 \log. a \dots\dots 2.365488 & & \\
 \log. \sin. C \dots 9.658037 & & \\
 \hline
 & 12.023525 & \\
 \log. \sin. A \dots 9.782796 & & \\
 \hline
 \log. c \dots\dots 2.240729 & & \\
 \therefore c = 174.07 & &
 \end{array}$$

EXAMPLES.

Find the other parts of the triangle ABC, having given :

149. $a = 178.3$	<i>Ans.</i> $A = 54^\circ 2'$, or $125^\circ 58'$
$b = 145$	$C = 84.48$, or 12.52
$B = 41^\circ 10'$	$c = 219.32$, or 49.05
150. $a = 2597.84$	„ $B = 80^\circ 39' 45''$, or $99^\circ 20' 15''$
$b = 3084.33$	$C = 43.730$, or 24.27
$A = 56^\circ 12' 45''$	$c = 2136.7$, or 1293.7

RULE III.

Two sides and the included angle being given, to find the remaining angles.

Subtract the given angle from 180° , and thus find the supplement: divide the supplement by 2. Put down the two given sides, and take their sum and difference.

To the log. tangent of half the supplement of the given angle, add the log. of the difference of the given sides; and from the result subtract the log. of the sum of the given sides; the remainder will be the log. tangent of half the difference of the required angles: which take from the tables.

To half the supplement of the given angle, add half the difference of required angles just found, and the sum will be the *greater* of the two angles required. To find the *less* of the two required angles, subtract the half difference.

NOTE. This case is usually worked by expressing the above

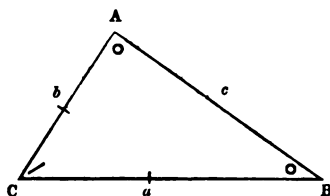
rule in the form of a proportion: thus, if a , b , and C be the given quantities, then (Part II.)

$$a+b : a-b :: \tan. \frac{1}{2} (A+B) : \tan. \frac{1}{2} (A-B).$$

The first three terms in this proportion are known; for half the sum of the required angles, or $\frac{1}{2} (A+B) = \frac{1}{2} (180^\circ - C)$ (the half supplement of given angle): hence the fourth term is easily found, which gives us half the difference of the required angles. The sum of the half sum and half difference of the required angles will be the angle opposite the greater of the two given sides; and their difference will be the less angle.

EXAMPLES.

Given $a = 798$, $b = 460$,
and $C = 55^\circ 2' 15''$: required
the angles A and B .



$$\begin{array}{rcl}
 a+b : a-b :: \tan. \frac{1}{2} (A+B) : \tan. \frac{1}{2} (A-B) & & \\
 a \dots 798 & \log. \tan. \frac{1}{2} (A+B) & 10.283138 \\
 b \dots 460 & \log. (a-b) \dots & 2.528917 \\
 \hline
 a+b \dots 1258 & & 12.812055 \\
 a-b \dots 338 & \log. (a+b) \dots & 3.099681 \\
 180^\circ 0' 0'' & \log. \tan. \frac{1}{2} (A-B) & 9.712374 \\
 55 \quad 2 \quad 15 \dots C & \therefore \frac{1}{2} A - \frac{1}{2} B = 27^\circ 16' 45'' & \\
 2) 124 \quad 57 \quad 45 \dots A+B & \text{and } \frac{1}{2} A + \frac{1}{2} B = 62 \quad 28 \quad 52 & \\
 \hline
 62 \quad 28 \quad 52 \dots \frac{1}{2} (A+B) & \therefore A = 89 \quad 45 \quad 37 & \\
 & \text{and } B = 35 \quad 12 \quad 7 &
 \end{array}$$

Find the other two angles of the triangle ABC, having given:

151.	$a=399$	<i>Ans.</i>	$A=89^\circ 45' 37''$
	$b=230$		$B=35 \quad 12 \quad 7$
	$C=55^\circ 2' 15''$		
152.	$b=64$	„	$B=52 \quad 54 \quad 30$
	$c=70$		$C=60 \quad 45 \quad 0$
	$A=66^\circ 20' 30''$		

153. $a=512$

$b=627$

$C=42^{\circ} 53' 38''$

Ans. $A=54^{\circ} 8' 11''$

$B=82 58 11$

RULE IV.—FIRST METHOD, USING HAVERSINES.*

Given two sides and the included angle, to find the third side.

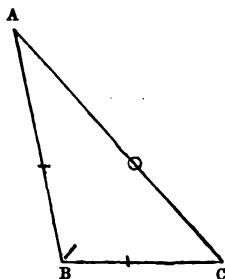
Add together $10\cdot602060$, log. haversine of given angle, and logarithm of each given side: take half the sum, from which subtract log. of difference of the given sides. Look in the tables for the remainder as a log. tangent, and take out the corresponding log. sine, which subtract from the half sum just used. The remainder will be the log. of the required side.

NOTE. When the half sum is first found, write it down again a little to the right hand, for the convenience of placing under it the log. sine to be subtracted.

EXAMPLES.

In the triangle ABC, given $a=20$, $c=30$, and $B=100^{\circ}$: required b .

constant log.	10·602060	
log. hav. B..	9·768508	
$a \dots 20$. log. a	1·301030	
$c \dots 30$. log. c	1·477121	
$a \sim c \dots 10$	<u>2)23·148719</u>	
	11·574359	 11·574359
log. ($a \sim c$) ..	<u>1·000000</u>	
log. tan.	10·574359	log. sin. .. 9·985104
		<u>log. b 1·589255</u>
		$b=38\cdot84$



* If the student have no table of haversines, he may work the examples in this rule by finding, in the first place, the two other angles by Rule III, and then the required side by Rule II; or he may proceed as follows: Instead of using log. haversine, as directed in the rule, take out twice the log. sine of half the given angle, and reject the 10 in the index of the constant log., then proceed as in the above rule; or he may find the side required by the next method.

RULE IV.—SECOND METHOD, WITHOUT HAVERSINES.

Given two sides and the included angle, to find the third side.

Add together the two given sides, and also subtract them : thus find their sum and difference. Put down also half the given angle.

Under heads (1) and (2) write down the following logarithms.

Under (1) and (2) the log. sum of given sides.

Under (1) the log. difference of given sides.

Take the difference of the logs. under (1). Again, under (1) put the log. tangent of half the given angle ; under (2) put the log. sine of half the given angle.

Add together the two logarithms under (2), and also the last two logarithms under (1).

Look out the result under (1) as a log. tangent, and take out the log. sine corresponding thereto, which subtract from the log. under (2). The result will be log. of required side.

EXAMPLE.

In the plane triangle ABC, given $a=178$, $b=145$, and $C=84^\circ 48'$: required the third side c .

	[1].	[2].
$a..178$	$\log. (a+b) ..2.509203$	$\log. (a+b) ..2.509203$
$b..145$	$\log. (a-b) ..1.518514$	$\log. \sin. \frac{1}{2} C ..9.828855$
$a+b..323$	difference $...0.990689$	sum $.....12.338058$
$a-b .. 33$	$\log. \tan. \frac{1}{2} C ..9.960530$	$\log. \sin. \text{arc} ..9.997299$
$\frac{1}{2} C = 42^\circ 24'$	$\log. \tan. \text{arc} .10.951219$	$\log. c 2.340759$
		$\therefore \text{side } c = 219.2$

Find the third side in the following examples :

154. $a=798$

$b=460$

$C=55^\circ 2' 16''$

Ans. $c=654$

155. $c=48$

$a=96$

$B=49^\circ 0'$

„ $b=73.98$

156. $a=512$

$b=627$

$C=42^{\circ} 53' 38''$

Ans. $c=430$

157. $b=2$

$c=2\frac{1}{2}$

$A=22^{\circ} 20' 0''$

„ $a=1$

RIGHT-ANGLED TRIANGLES.

Right-angled triangles may be solved by the rules already given; but the easiest way is to make use of the trigonometrical ratios or definitions for the sine, tangent, &c. given in Art. (5), p. 3, as in the following rule.

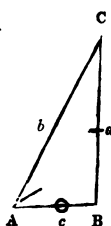
RULE V.

Right-angled Plane Triangles.

Select that expression or ratio in p. 3 which contains the two known terms and also the term required, and when possible let the numerator of the ratio or fraction selected be the part required. Reduce it to tabular logarithms, by Art. (32), p. 28, and then find the value of the unknown term in the usual manner. See Ex. 122, p. 29.

EXAMPLES.

1. In the right-angled plane triangle ABC, given $B=90^{\circ}$, $a=42$, and $A=50^{\circ} 10'$: required the other parts.



Mark the given parts, namely A and a , with a stroke of the pen; and if c is the part about to be found, mark it with a different character, as in the figure. Then

[1]. To find c .

By p. 3, ratio (d), $\frac{c}{a} = \cot. A$, or $c = a \cot. A$

$$\therefore \log c = \log. a + \log. \cot. A - 10$$

Calculation of c.

$\alpha = 42$	$\log. a \dots\dots 1.623249$
$A = 50^\circ 10'$	$\log. \cot. A \dots 9.921247$
	$\log. c \dots\dots 1.544496$
	$\therefore c = 35.03$

[2]. To find b .

Draw the triangle again; mark the given parts A and α with a stroke, and b , the part required, with a different mark as before: then by ratio (b), p. 3, $\frac{b}{a} = \operatorname{cosec}. A$, or $b = a \operatorname{cosec}. A$

$\therefore \log. b = \log. a + \log. \operatorname{cosec}. A - 10$	
$\alpha = 42$	$\log. a \dots\dots 1.623249$
$A = 50^\circ 10'$	$\log. \operatorname{cosec}. A \dots 10.114689$
	$\log. b \dots\dots 1.737938$
	$\therefore b = 54.7$

To find C : $C = 90^\circ - A = 39^\circ 50'$

2. Given $\alpha = .02$, $c = 0.1$, and $B = 90^\circ$: required the other parts. Make a triangle ABC , right-angled at B (see last fig.); mark the given sides α and c with a stroke of the pen, mark also the part required with a different character.

Let the part required be A .[1]. To find A .By p. 3, ratio (c), $\tan. A = \frac{\alpha}{c}$

$$\therefore \log. \tan. A - 10 = \log. \alpha - \log. c$$

$$\text{or } \log. \tan. A = 10 + \log. \alpha - \log. c$$

Calculation.

$\alpha = .02$	$\log. a \dots\dots 2.301030$
$c = .1$	$\log. c \dots\dots 1.000000$
	$\log. \tan. A \dots 9.301030$
	$\therefore A = 11^\circ 18' 30''$

[2]. To find side b .

By p. 3, ratio (e) , $\frac{b}{c} = \sec. A$

$$\therefore b = c \sec. A$$

$$\log. b = \log. c + \log. \sec. A - 10$$

$c = 1$	$\log. c \dots\dots$	$\bar{1} \cdot 000000$
$A = 11^\circ 18' 30''$	$\log. \sec. A \dots$	$10 \cdot 008514$
		$9 \cdot 008514$
		10
	$\log. b \dots\dots$	$\bar{1} \cdot 008514$
	$\therefore b =$	$1 \cdot 02$

To find B : $B = 90^\circ - A = 78^\circ 41' 30''$

When two sides of a right-angled triangle are given, the third may be found by the well-known property of right-angled plane triangles, namely, "the square of the side opposite the right angle is equal to the sum of the squares of the sides containing the right angle" (Euc. i. 47): thus, in a triangle ABC , right-angled at C , $a^2 + b^2 = c^2$; and if $a = \cdot 02$, and $b = \cdot 1$, we have

$$c^2 = (\cdot 02)^2 + (\cdot 1)^2, \text{ or } c = \sqrt{0104} = \cdot 102 \text{ nearly.}$$

If one of the given sides is the hypotenuse, this method of finding the third side may be simplified: thus, in the above example, given $c = \cdot 102$, and $a = \cdot 02$; to find b .

$$\text{Since } a^2 + b^2 = c^2$$

$$\therefore b^2 = c^2 - a^2 = (c+a) \cdot (c-a) = \cdot 122 \times \cdot 082 = \cdot 010004$$

$$\therefore b = \cdot 1 \text{ nearly.}$$

Find the other parts of the right-angled triangle ABC , having given

158.	$A = 52^\circ 38' 0''$	<i>Ans.</i>	$C = 37^\circ 22'$
	$b = 45$		$a = 35 \cdot 76$
	$B = 90^\circ$		$c = 27 \cdot 31$
159.	$A = 49^\circ 14'$	„	$C = 40^\circ 46'$
	$c = 331$		$a = 384$
	$B = 90^\circ$		$b = 506 \cdot 8$

160.	$A=56^{\circ} 29' 15''$ $b=4264.3$ $B=90^{\circ}$	<i>Ans.</i>	$a=3555$ $c=2354$ $C=33^{\circ} 30' 45''$
161.	$A=4^{\circ} 44'$ $a=694.73$ $B=90^{\circ}$	,,	$c=8390$ $b=8419$ $C=85^{\circ} 16'$
162.	$b=.2$ $A=40^{\circ}$ $B=90^{\circ}$	,,	$a=.1286$ $c=.1532$ $C=50^{\circ}$
163.	$c=.04$ $C=40^{\circ}$ $B=90^{\circ}$	,,	$a=.04767$ $b=.06223$ $A=50^{\circ}$
164.	$a=1777.5$ $c=1177$ $B=90^{\circ}$	,,	$A=56^{\circ} 29' 15''$ $C=33^{\circ} 30' 45''$ $b=2132.1$

RULE VI.

Two sides and the included angle being given, to find the area.

Add together the logarithms of the two given sides, and log. sine of the given angle; the sum, rejecting 10 in the index, will be the logarithm of twice the required area.

EXAMPLE,

Given $a=798$, $b=460$, and $C=55^{\circ} 2' 15''$: required the area.

log. a		2.902003
log. b		2.662758
log. sin. C		9.913563
log. 2 area		5.478324
$\therefore$ 2 area		300832
and area		<u>150416</u>

165. Given $a=245$ yards, $b=760$ yards, and $C=60^{\circ}$: required the area.

Ans. 80627 square yards.

RULE VII.

Three sides of a plane triangle being given, to find the area.

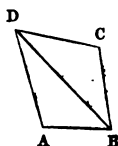
From half the sum of the three sides, subtract each side separately. Add together the log. of the half sum and the logarithms of the three remainders. Half the result will be the logarithm of the area.

EXAMPLE.

166. Given $a=798$, $b=460$, and $c=654$: required the area.

$a \dots 798$				log. 956 .. 2.980458
$b \dots 460$	956	956	956	log. 158 .. 2.198657
$c \dots 654$	798	460	654	log. 496 .. 2.695482
$2)1912$	158	496	302	log. 302 .. 2.480007
956				$2)10.354604$
				5.177302

$\therefore \text{area} = 150418$



167. In the trapezium ABCD, $AB=90$ yards, $BC=100$ yards, $CD=110$ yards, $DA=120$ yards, the diagonal $BD=178.8$ yards: required the area of the trapezium.

Ans. 9768.7 square yards.

SPHERICAL TRIGONOMETRY.

CHAPTER III.

RULES IN SPHERICAL TRIGONOMETRY.

RULE VIII.—FIRST METHOD, USING HAVERSINES.

Three sides of a spherical triangle being given, to find an angle.*

Put down the two sides containing the required angle, and take the difference, under which put the third side ; take the sum and difference.

Add together the log. cosecants of the two first terms in this form (rejecting the tens from the index), and the *halves*† of the log. haversines of the two last terms. The result will be the log. haversine of the required angle.

RULE VIII.—SECOND METHOD, WITHOUT HAVERSINES.

Three sides of a spherical triangle being given, to find an angle.

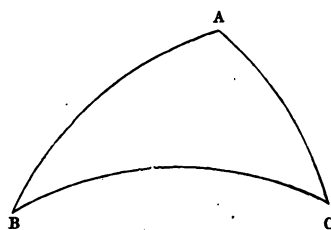
Put down the two sides containing the required angle, and

* A *spherical* triangle is that part of the surface of a sphere which is bounded by arcs of three *great* circles, that is, three circles whose planes pass through the center of the sphere. The three arcs are the *sides* of the triangle ; and any one of its *angles* is the same as the inclination of the planes of the sides containing the angle. For other definitions in Spherical Trigonometry, see *Trigonometry*, Part II.

† With very little practice, the halves of the log. haversines may be taken out of the table by *inspection*.

take the difference ; under which put the third side : take the sum and difference, and the half sum and half difference.

Add together the log. cosecants of the two first terms in



this form (rejecting the tens from the index), and the log. sines of the two last terms ; divide by 2, and look out the result as a log. sine ; the angle corresponding to which will be half the required angle.

168. In the spherical triangle ABC, given $a=124^{\circ} 10' 0''$, $b=89^{\circ} 0' 15''$, $c=108^{\circ} 40' 0''$: required the angle A.

By First Method.

$89^{\circ} 0' 15''$	log. cosec.	0.000066
$108\ 40\ 0$	log. cosec.	0.023468
<u>$19\ 39\ 45$</u>	$\frac{1}{2}$ log. hav. S	4.978000
$124\ 10\ 0$	$\frac{1}{2}$ log. hav. D	4.898018
S .. $143\ 49\ 45$	log. hav. A	9.899552
D .. $104\ 30\ 15$		$\therefore A=125^{\circ} 56' 45''$

By Second Method.

$89^{\circ} 0' 15''$	log. cosec.	0.000066
$108\ 40\ 0$	log. cosec.	0.023468
<u>$19\ 39\ 45$</u>	log. sin. ($\frac{1}{2}$ S)	9.978000
$124\ 10\ 0$	log. sin. ($\frac{1}{2}$ D)	9.898018
S .. $143\ 49\ 45$		2) <u>19.899552</u>
D .. $104\ 30\ 15$	log. sin. $\frac{1}{2}$ A ..	9.949776
$\frac{1}{2}$ S .. $71\ 54\ 52$		$\therefore \frac{1}{2} A=62^{\circ} 58' 15''$
$\frac{1}{2}$ D .. $52\ 15\ 7$		<u>2</u>
	and $A=$	<u>$125\ 56\ 30$</u>

EXAMPLES.

Find the three angles of the spherical triangle ABC, having given :

169.	$a = 49^\circ 10' 0''$	<i>Ans.</i>	$A = 59^\circ 2' 0''$
	$b = 58 \ 25 \ 0$		$B = 74 \ 54 \ 0$
	$c = 56 \ 42 \ 0$		$C = 71 \ 18 \ 30$
170.	$a = 119 \ 42 \ 20$	„	$A = 115 \ 38 \ 45$
	$b = 108 \ 4 \ 18$		$B = 99 \ 21 \ 15$
	$c = 68 \ 53 \ 42$		$C = 75 \ 31 \ 30$
171.	$a = 87 \ 10 \ 15$	„	$A = 81 \ 24 \ 0$
	$b = 62 \ 36 \ 45$		$B = 61 \ 31 \ 15$
	$c = 100 \ 10 \ 15$		$C = 102 \ 59 \ 0$

RULE IX.—FIRST METHOD, USING HAVERSINES.

Two sides and the included angle being given, to find the remaining side.

Add together 6·301030, the log. sines of the given sides, and log. haversine of given angle; reject 30 from the index of the sum, and take out the natural number of the resulting logarithm. To this natural number add the natural versine of the difference of the given sides; the sum will be the natural versine of the required side, which find from the tables.

RULE IX.—SECOND METHOD, WITHOUT HAVERSINES.

Two sides and the included angle being given, to find the remaining side.

Add together 6·301030, the log. sines of the given sides, and twice the log. sine of half the given angle, rejecting 40 from the index of the sum; and take out the natural number of the resulting logarithm. To this natural number add the natural versine of the difference of the given sides; the sum will be the natural versine of the required side, which find from the tables.

RULE IX.—THIRD METHOD (REQUIRING ONLY THE COMMON TABLE OF SINES, &c.).

Given two sides and the included angle, to find the third side.

Take the difference between the two given sides: divide

it by 2, and thus get half the difference of given sides : find also half the given angle.

Add together the log. sines of the two given sides, and divide the result by 2, to which add the log. sine of half the given angle. Call this sum log. M. From log. M subtract log. sine of half the difference of given sides, and look in the tables for this result as a log. tangent; and take out the corresponding log. sine, which subtract from log. M. The remainder will be the log. sine of half the required side.

NOTE. When log. M is first found, write it down again a little to the right hand, for the convenience of placing under it the log. sine to be subtracted.

EXAMPLE.

In the spherical triangle ABC, given $c=119^\circ 42' 20''$, $b=108^\circ 4' 18''$, and $A=75^\circ 31' 30''$; required the remaining side a .

By First Method.

	const. log. ..	6.301030
$c=119^\circ 42' 20''$	log. sin. c ..	9.938818
$b=108 \quad 4 \quad 18$	„ sin. b ..	9.978031
$c-b=11 \quad 38 \quad 2$	„ hav. A ..	9.574056
	log.	5.791935
nat. No.	619349	
nat. ver. $(c-b)$..	20544	
nat. ver. a	639893	
$\therefore a=68^\circ 53' 36''$		

By Second Method.

	const. log. ..	6.301030
$c=119^\circ 42' 20''$	log. sin. c ..	9.938818
$b=108 \quad 4 \quad 18$	„ sin. b ..	9.978031
$c-b=11 \quad 38 \quad 2$	„ sin. $\frac{1}{2}A$..	9.787028
$\frac{1}{2}A=37 \quad 45 \quad 45$	„ sin. $\frac{1}{2}A$..	9.787028
	log.	5.791935

$$\begin{array}{rcl}
 \text{nat. No.} & \dots\dots\dots & 619349 \\
 \text{nat. ver. } (c-b) & \dots & 20544 \\
 \text{nat. ver. } a & \dots\dots\dots & \underline{639893} \\
 \therefore a & = & 68^\circ 53' 36''
 \end{array}$$

By Third Method.

$$\begin{array}{rcl}
 b=108^\circ 4' 18'' & \log.\sin.b & \dots 9.978031 \\
 c=119 42 20 & \log.\sin.c & \dots 9.938818 \\
 2) \overline{11\ 38\ 2} & & 2) \overline{19.916849} \\
 \frac{1}{2}(b \sim c) & \overline{5\ 49\ 1} & \quad 9.958424 \\
 & \log.\sin.\frac{1}{2}A & \dots 9.787028 \\
 \frac{1}{2}A = 37\ 45\ 45 & \log.M & \dots 19.745452 \dots\dots 19.745452 \\
 & \log.\sin.\frac{1}{2}(b \sim c) & \dots 9.005805 \\
 & \log.\tan.\dots \overline{10.739647} & \log.\sin.\dots 9.992916 \\
 & & \log.\sin.\frac{1}{2}a \dots 9.752536 \\
 & & \therefore \frac{1}{2}a = 34^\circ 26' 45'' \\
 & & \quad \quad \quad 2 \\
 & & \therefore \text{side } a = \underline{68\ 53\ 30}
 \end{array}$$

EXAMPLES.

Find the third side of the spherical triangle ABC, having given:

- | | | |
|------|------------------------|------------------------------------|
| 172. | $A = 96^\circ 32' 0''$ | |
| | $b = 76\ 42\ 0$ | |
| | $c = 89\ 10\ 30$ | <i>Ans.</i> $a = 96^\circ 9' 57''$ |
| 173. | $A = 50\ 0\ 0$ | |
| | $b = 70\ 45\ 10$ | |
| | $c = 62\ 10\ 15$ | „ $a = 46\ 19\ 32$ |
| 174. | $a = 100\ 8\ 42$ | |
| | $b = 98\ 10\ 5$ | |
| | $C = 88\ 24\ 24$ | „ $c = 87\ 0\ 50$ |
| 175. | $b = 118\ 2\ 14$ | |
| | $c = 120\ 18\ 33$ | |
| | $A = 27\ 22\ 34$ | „ $a = 23\ 57\ 9$ |

$$176. \quad a = 87^{\circ} 10' 15''$$

$$b = 62 \ 36 \ 45$$

$$C = 102 \ 58 \ 30$$

$$\text{Ans. } c = 100^{\circ} \ 9' \ 38''$$

$$177. \quad a = 69 \ 19 \ 10$$

$$b = 78 \ 59 \ 14$$

$$C = 110 \ 48 \ 42$$

$$c = 104 \ 59 \ 57$$

RULE X.

Of two sides and the opposite angles, any three being given, to find the fourth.

Write down a proportion having for the two first terms the sines of the two sides concerned, and for the third term the sine of the angle opposite to the first side put down, and for the fourth term the sine of the angle opposite to the other side put down in the proportion. Mark the term required, and proceed as in the corresponding rule for plane triangles (p. 41).

EXAMPLES.

[1]. In the spherical triangle ABC, given $a = 70^{\circ} 10' 30''$, $b = 80^{\circ} 5'$, and $B = 33^{\circ} 15'$; required the angle A.

[2]. $b = 119^{\circ} 42' 20''$, $a = 108^{\circ} 4' 18''$, and $A = 99^{\circ} 21' 30''$; required B.

$$\sin. a : \sin. b :: \sin. A : \sin. B$$

$$\sin. a : \sin. b :: \sin. A : \sin. B$$

$$\log. \sin. a \dots 9.973466$$

$$\log. \sin. b \dots 9.938818$$

$$,, \sin. B \dots 9.739013$$

$$,, \sin. A \dots 9.994181$$

$$19.712479$$

$$19.932999$$

$$,, \sin. b \dots 9.993462$$

$$,, \sin. a \dots 9.978031$$

$$,, \sin. A \dots 9.719017$$

$$,, \sin. B \dots 9.954968$$

$$\therefore A = 31^{\circ} 34' 30''$$

$$\therefore B = 180^{\circ} - 64^{\circ} 21' 30''$$

$$= 115^{\circ} 38' 30''$$

In example [1], a being less than b , A is less than B , and therefore A is less than 90° .

In example [2], b being greater than a , B is greater than A , and therefore B is greater than 90° : hence the angle taken out of the tables must be subtracted from 180° to get B .

Unless by some limitation of this kind it should be de-

terminated that the required angle is less or greater than 90° , this rule is considered ambiguous. See corresponding rule for plane triangles (p. 44).

RULE XI.

Two sides and the included angle being given, to find the other two angles.

Take the sum and difference of the two given sides, and the half sum and half difference, and also half the given angle.

Under heads [1] and [2], put down the following logarithms:

Under [1] and [2], log. cot. of half the given angle.

Under [1], log. cosec. } of half the sum of the given sides

Under [2], log. sec. } (rejecting 10 in each index).

Under [1], log. sin. } of half the difference of given sides.

Under [2], log. cos. }

Add together the logarithms under [1] and [2], and (rejecting 10 from each index) the results will be the log. tangents of half the difference and half the sum of the required angles respectively, which take from the tables (a).

The sum of the two arcs thus found will be the angle opposite to the greater of the given sides; and their difference will be the angle opposite to the less of the given sides.

(a) NOTE. If half the sum of the given sides be greater than 90° , half the sum of the required angles will also be greater than 90° : in this case, the arc found as above under [2] must be subtracted from 180° to get half the sum of the required angles.

EXAMPLE.

178. In the spherical triangle ABC, given $a=124^\circ 10'$, $b=89^\circ 0' 15''$, $C=112^\circ 1' 30''$: required the other two angles, A and B.

$$\begin{array}{rcl}
 a & = & 124^\circ 10' 0'' \\
 b & = & 89 \quad 0 \quad 15 \\
 a+b & = & 213 \quad 10 \quad 15 \dots\dots\dots \frac{1}{2}(a+b) = 106 \quad 35 \quad 7 \\
 a-b & = & 35 \quad 9 \quad 45 \dots\dots\dots \frac{1}{2}(a-b) = 17 \quad 34 \quad 52
 \end{array}
 \qquad
 \begin{array}{rcl}
 C & = & 112^\circ 1' 30'' \\
 \therefore \frac{1}{2} C & = & 56 \quad 0 \quad 45
 \end{array}$$

[1].

$$\begin{aligned}
 \cot. \frac{1}{2} C & \dots\dots 9.828783 \\
 \operatorname{cosec}. \frac{1}{2} (a+b) & \dots 0.018455 \\
 \sin. \frac{1}{2} (a-b) & \dots 9.480090 \\
 \tan. \frac{1}{2} (A-B) & \dots 9.327328 \\
 \therefore \frac{1}{2} (A-B) & \dots 11^\circ 59' 45''
 \end{aligned}$$

[2].

$$\begin{aligned}
 \cot. \frac{1}{2} C & \dots\dots 9.828783 \\
 \sec. \frac{1}{2} (a+b) & \dots 0.544480 \\
 \cos. \frac{1}{2} (a-b) & \dots 9.979225 \\
 \tan. \operatorname{arc} & \dots\dots 10.352488 \\
 \therefore \operatorname{arc} & \dots 66^\circ 3' 0''^*
 \end{aligned}$$

$$\begin{array}{r}
 180 \\
 \hline
 \therefore \frac{1}{2} (A+B) \dots 113 \ 57 \ 0 \\
 \text{and } \frac{1}{2} (A-B) \dots 11 \ 59 \ 45 \\
 \hline
 \therefore A = 125 \ 56 \ 45 \\
 B = 101 \ 57 \ 15
 \end{array}$$

179. Given $b=89^\circ 0' 15''$, $c=108^\circ 40'$, and $A=125^\circ 56' 45''$; to find the angles B and C.

Ans. $B=101^\circ 57' 15''$, $C=112^\circ 2'$.

Examples to this and the following rule are easily formed out of those already given; as Ex. 169, &c.

RULE XII.

Given two angles and the included side, to find the other two sides.

Take the sum and difference of the two given angles, and the half sum and half difference, and also half the given side.

Under heads [1] and [2] put down the following logarithms:

Under [1] and [2], log. tan. of half the given side.

Under [1], log. cosec. } of half the sum of the given angles
Under [2], log. sec. } (rejecting 10 in each index).

Under [1], log. sin. } of half the difference of given angles.
Under [2], log. cos. }

Add together logarithms under [1] and [2], and (rejecting 10 in each index) the results will be log. tangents of half the difference and half the sum of required sides respectively (a). The sum of the two arcs will be the side opposite the greater

* The arc $66^\circ 3'$ is subtracted from 180° , because the sum of the sides a and b is greater than 180° . (See note, p. 61.)

of the given angles, and their difference will be the side opposite to the less of the given angles.

(a) NOTE. If half the sum of the given angles exceed 90° , the arc taken out under [2] must be subtracted from 180° , to give half the sum of the required sides.

EXAMPLE.

180. In the spherical triangle ABC, given $A = 113^\circ 33' 30''$, $B = 51^\circ 30' 20''$, and $c = 60^\circ 18' 4''$; required the other two sides.

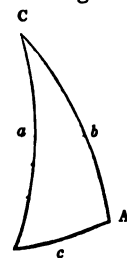
$A = 113^\circ 33' 30''$	$c = 60^\circ 18' 4''$
$B = 51^\circ 30' 20''$	$\therefore \frac{1}{2} c = 30^\circ 9' 2''$
$A + B = 165^\circ 30' 50''$	$\frac{1}{2} (A + B) = 82^\circ 31' 55''$
$A - B = 62^\circ 3' 10''$	$\frac{1}{2} (A - B) = 31^\circ 1' 35''$
[1].	[2].
$\tan. \frac{1}{2} c \dots\dots 9.764070$	$\tan. \frac{1}{2} c \dots\dots 9.764070$
$\operatorname{cosec}. \frac{1}{2} (A + B) 0.003700$	$\sec. \frac{1}{2} (A + B) 0.886220$
$\sin. \frac{1}{2} (A - B) \dots 9.712173$	$\cos. \frac{1}{2} (A - B) 9.932952$
$\tan. \frac{1}{2} (a - b) \dots 9.479943$	$\tan. \frac{1}{2} (a + b) \dots 10.583242$
$\frac{1}{2} (a - b) \dots\dots 16^\circ 48'$	$\frac{1}{2} (a + b) \dots\dots 75^\circ 22'$
	$\frac{1}{2} (a - b) \dots\dots 16^\circ 48'$
	$a = 92^\circ 10'$
	$b = 58^\circ 34'$

RULE XIII.

RIGHT-ANGLED SPHERICAL TRIANGLES.

In a right-angled spherical triangle ABC, A being the right angle, the two sides b and c , and the complements* of the three other parts, namely co. a , co. B , and co. C , are called the *five circular parts*. By the following rules, called, from the name of their author, *Napier's Rules*, two of these parts being given, any one of the other three may be found.

In applying the rules, we must select one of the three parts concerned, such, that the other



* The complement of an angle is what it wants of 90° ; thus, in the

two may either be *both adjacent* to it, or *both separate* from it. The part so selected is called the *middle part*. Rule A will apply to the former case, Rule B to the latter.

RULE A.

The *sine* of the middle part is equal to the product of the *tangents* of the two parts *adjacent* to it.

RULE B.

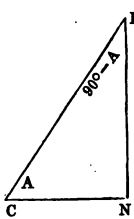
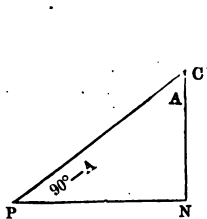
The *sine* of the middle part is equal to the product of the *cosines* of the two parts *opposite* to, or *separated* from it.

Having written down the equation according to the case, make a dash under the part required, and determine its magnitude by applying the proper signs (+ or -) to each term (Rule XV. p. 31).

EXAMPLES.

In a right-angled spherical triangle ABC, given $b = 74^{\circ} 19' 30''$, $c = 38^{\circ} 56'$, and $A = 90^{\circ}$; required the other three parts.

Make a triangle for each of the three required parts, and



right-angled triangle PNC (1st fig.), the angle P, or $90^{\circ} - A$, is the complement of the angle A. It may easily be proved that the sin., tan., sec., cos., cot., and cosec. of an angle are the cos., cot., cosec., sin., tan., and sec. of its complement respectively. Thus,

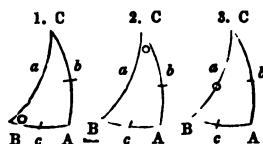
let the two triangles CPN be equal to each other in every respect, then P is the complement of A, or co. A.

$$\text{Now [Art. (5), p. 3] } \sin. A = \frac{PN}{CP} = \cos. (90^{\circ} - A) = \cos. \text{ co. A}$$

$$\tan. A = \frac{PN}{CN} = \cot. (90^{\circ} - A) = \cot. \text{ co. A, \&c.}$$

Hence, in the above rules, sin. A is substituted for cos. co. A ; cos. A for sin. co. A ; cot. A for tan. co. A, &c. (See Part II.)

mark the two given parts and the unknown part as directed (p. 42), and then consider which of the three is the middle part, and whether the other two marked parts are adjacent, or opposite parts.



Thus in fig. 1, to find B, the right angle A not being considered as a part, c will be the middle part, and complement B and b are the *adjacent* parts.

$$\therefore \text{By Rule A, } \sin. c = \tan. co. B \cdot \tan. b$$

$$\text{or } \sin. c = \cot. B \cdot \tan. b$$

(since $\tan. co. B = \cot. B$ by note, p. 64); and determining the sign of B by Rule XV. p. 31, we have

$$\begin{array}{ccc} + & + & + \\ \sin. c = & \cot. B \cdot \tan. b \end{array}$$

In logarithms,

$$\log. \sin. c - 10 = \log. \cot. B - 10 + \log. \tan. b - 10$$

$$\text{whence } \log. \cot. B = \log. \sin. c + 10 - \log. \tan. b$$

Calculation.

$$\begin{array}{ll} b = 74^\circ 19' 30'' & \log. \sin. c + 10 \dots 19.798247 \\ c = 38 \ 56 & \text{,, } \tan. b \dots\dots\dots 10.551887 \\ & \text{,, } \cot. B \dots\dots\dots 9.246360 \\ & \therefore B = 80^\circ \end{array}$$

In fig. 2, to find C. Marking the figure in the usual manner, we see that b is the middle part, and c and C the *adjacent* parts.

$\therefore$ By Rule A,

Calculation.

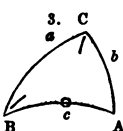
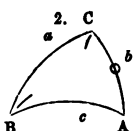
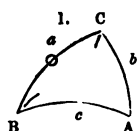
$$\begin{array}{ll} \sin. b = \tan. c \cdot \tan. co. C & \log. \sin. b + 10 \dots 19.983540 \\ + & + \\ \text{or } \sin. b = \tan. c \cdot \cot. C & \text{,, } \tan. c \dots\dots\dots 9.907336 \\ & \text{,, } \cot. C \dots\dots\dots 10.076204 \\ & \therefore C = 40^\circ \end{array}$$

In fig. 3, to find a . In this case a is the middle part, and b and c are the parts separated from it.

∴ By Rule B,

Calculation.

$$\begin{array}{rcl}
 \sin. co. a = \cos. b. \cos. c & \log. \cos. b & \dots\dots 9.431654 \\
 + & + & \\
 + & + & \\
 \text{or } \cos. a = \cos. b. \cos. c & \log. \cos. a \text{ (rejecting 10)} & \underline{9.322565} \\
 & \therefore a = 77^\circ 52'
 \end{array}$$



Given $A = 90^\circ$, $B = 78^\circ 10'$, and $C = 131^\circ 32' 45''$; to find the other parts.

To find a (fig. 1).

$$\begin{array}{rcl}
 - & + & - \\
 \cos. a = \cot. B. \cot. C & \log. \cot. B & 9.321222 \\
 & \text{,, } \cot. C & 9.947508 \\
 & \text{,, } \cos. a & 9.268730 \\
 & & 79^\circ 18' 0'' \\
 & \frac{180}{\therefore a = 100} & 42 \ 0
 \end{array}$$

To find b (fig. 2).

$$\begin{array}{rcl}
 + & + & + \\
 \cos. B = \cos. b. \sin. C & \cos. B + 10 & 19.311893 \\
 & \sin. C & \dots\dots 9.874148 \\
 & \cos. b & \dots\dots 9.437745 \\
 & \therefore b = 74^\circ & 5' 45''
 \end{array}$$

To find c (fig. 3).

$$\begin{array}{rcl}
 - & + & - \\
 \cos. C = \sin. B. \cos. c & \cos. C + 10 & 19.821657 \\
 & \sin. B & \dots\dots 9.990671 \\
 & \cos. c & \dots\dots 9.830986 \\
 & & 47^\circ 20' 30'' \\
 & \frac{180}{\therefore c = 132} & 39 \ 30
 \end{array}$$

In a right-angled spherical triangle ABC, find the other parts, having given :

181.	$B = 72^\circ 19' 0''$	$Ans. a = 54^\circ 28' 0''$
	$b = 50 \ 50 \ 0$	$c = 23 \ 2 \ 15$
	$A = 90 \ 0 \ 0$	$C = 28 \ 45 \ 0$
182.	$b = 60 \ 10 \ 0$	,, $B = 60 \ 32 \ 45$
	$c = 100 \ 0 \ 0$	$C = 98 \ 41 \ 45$
	$A = 90 \ 0 \ 0$	$a = 94 \ 57 \ 15$
183.	$B = 100 \ 0 \ 0$	,, $a = 90 \ 30 \ 0$
	$C = 87 \ 10 \ 0$	$b = 100 \ 0 \ 45$
	$A = 90 \ 0 \ 0$	$c = 87 \ 7 \ 15$
184.	$c = 46 \ 18 \ 23$	,, $b = 26 \ 23 \ 15$
	$B = 34 \ 27 \ 30$	$a = 51 \ 46 \ 15$
	$A = 90 \ 0 \ 0$	$C = 66 \ 59 \ 30$
185.	$c = 118 \ 21 \ 4$	,, $b = 116 \ 18 \ 0$
	$A = 23 \ 40 \ 12$	$C = 100 \ 59 \ 30$
	$B = 90 \ 0 \ 0$	$a = 21 \ 5 \ 45$

186.	$a=100^{\circ} 42' 0''$	<i>Ans.</i>	$C=131^{\circ} 32' 45''$
	$B=78 10 0$		$c=132 39 30$
	$A=90 0 0$		$b=74 5 45$
187.	$c=53 14 20$	"	$b=91 4 15$
	$A=91 25 58$		$C=53 15 0$
	$B=90 0 0$		$a=91 47 15$

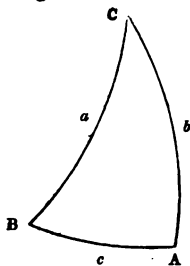
RULE XIV.

QUADRANTAL SPHERICAL TRIANGLES.

A spherical triangle having one *side* 90° is called a *quadrantal triangle*.

The five circular parts in a quadrantal triangle are the two angles adjacent to the quadrant, and the complements of the other three; thus, if ABC be a quadrantal triangle, the side a being 90° , the five circular parts are the angles B and C , and the complements of the angle A and of the sides b and c .

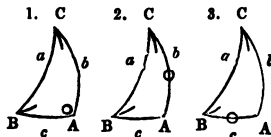
Any two of these parts being given, a third may be found by the preceding rules for right-angled triangles (p. 64).



The magnitude (whether greater or less than 90°) of the part required is determined by Rule XV. p. 31, observing, that *when two angles or two sides come together on the same side of the equation*, the sign — must be placed before them, and the three signs thus placed on one side of the equation must be made to produce the same result (positive or negative) as the sign of the other side.

EXAMPLES.

In the quadrantal triangle ABC , given $B=80^{\circ} 10'$, $C=48^{\circ} 50'$, and $a=90^{\circ}$; find the other parts.



To find A (fig. 1).	To find b (fig. 2).	To find c (fig. 3).
$\cos. A = -\cos. B. \cos. C$	$\sin. C = \tan. B. \cot. b$	$\sin. B = \tan. C. \cot. c$
log. cos. B . 9.232444	sin. C + 10 . 19.876678	sin. B + 10 . 19.993572
„ cos. C . 9.818392	tan. B . . . 10.761128	tan. C . . . 10.058287
„ cos. A . 9.050836	cot. b 9.115550	cot. c . . . 9.935285
83° 32' 45"	b = 82° 34'	c = 49° 15' 51"
180		
A = 96 27 15		

In a quadrantal spherical triangle ABC, find the other parts, having given :

188.	a = 90° 0' 0"	Ans.	B = 74° 36' 30"
	A = 100 0 0		b = 78 14 30
	c = 50 10 0		C = 49 8 0
189.	B = 45 0 0	„	A = 107 10 15
	c = 72 0 0		b = 47 44 30
	a = 90 0 0		C = 65 19 15
190.	b = 90 0 0	„	B = 88 36 45
	a = 100 0 0		A = 100 5 45
	c = 82 10 30		C = 82 3 15
191.	a = 90 0 0	„	A = 96 17 45
	B = 80 10 0		b = 82 26 0
	C = 50 2 0		c = 50 27 0
192.	A = 72 49 45	„	C = 114 40 45
	b = 47 44 30		B = 44 59 45
	a = 90 0 0		c = 108 0 0
193.	c = 49 23 45	„	A = 101 42 15
	b = 76 41 0		B = 72 20 30
	a = 90 0 0		C = 48 1 50
194.	a = 60 10 15	„	A = 59 41 45
	b = 80 20 30		B = 78 51 0
	c = 90 0 0		C = 95 36 0

EXAMINATION PAPERS IN PRECEDING RULES.

I.

195. Required the product of $18 \times 48 \times 6 \cdot 2 \times 4$ *Ans.* 21427

196. Divide 236 by 16·1 „ 14 66

197. Required the 3d power of 12·5 „ 1953·125

198. „ 4th power of 0·76543 „ 000034326

199. „ 6th root of 11078 „ 4·72146

200. „ 7th root of 0·98674 „ 0·7183146

201. „ value of $(19)^{\frac{2}{3}}$ „ 3·247202. „ value of x in the following proportion:

$$24 : 17 \cdot 5 :: 79 : x \quad \text{Ans. } x = 57 \cdot 604$$

203. Reduce the following expression to logarithms:

$$x = \frac{a^2 b}{cd} \quad \text{Ans. } \log. x = 2 \log. a + \log. b - \log. c - \log. d$$

204. Find the value of x in the following equation:

$$2^x = 769. \quad \text{Ans. } x = 9 \cdot 5868$$

205. The first term of a geometrical series is 2, its common ratio 3, and number of terms 8; find the sum.

$$\text{Ans. } S = 6560$$

206. If 400*l.* be placed out at compound interest for nine years, at 4*l.* per cent per annum, required its amount.

$$\text{Ans. } 569*l.* 6*s.* 8*d.*$$

II.

207. In the plane triangle ABC, given $a=10$, $b=15$, and $c=20$; required the angle A. *Ans.* $A=28^\circ 57' 15''$ 208. In the plane triangle ABC, given $a=25 \cdot 125$, $b=12 \frac{1}{2}$, and $A=68^\circ 48'$; to find B. *Ans.* $B=27^\circ 38' 15''$ 209. Given $a=399$, $b=230$, and $C=55^\circ 2' 15''$; required the angle A. *Ans.* $A=89^\circ 45' 37''$ 210. Given $a=40$, $b=60$, and $C=100^\circ$; required c .

$$\text{Ans. } c=77 \cdot 68$$

211. In the right-angled triangle ABC, given $B=90^\circ$, $a=210$, and $A=50^\circ 10'$; required the other parts.

$$\text{Ans. } C=39^\circ 50', c=175 \cdot 1, b=273 \cdot 5$$

212. Given $a=399$, $b=230$, and $C=124^{\circ} 57' 45''$; required the area. *Ans.* 37604

213. Given $a=399$, $b=230$, and $c=327$; required the area. *Ans.* 37604.7

214. Give definitions of the sine, cosine, tangent, cotangent, secant, cosecant, and versine of an angle.

III.

215. In the spherical triangle ABC, given $a=120^{\circ} 54'$, $b=105^{\circ} 6'$, and $c=108^{\circ} 41' 30''$; required the angles A, B, and C. *Ans.* $A=130^{\circ} 50'$, $B=121^{\circ} 35'$, and $C=123^{\circ} 18'$

216. In the spherical triangle ABC, given $a=64^{\circ} 21' 15''$, $b=80^{\circ} 38' 45''$, and $c=104^{\circ} 28' 30''$; required the angles A, B, and C.

Ans. $A=60^{\circ} 17' 45''$, $B=71^{\circ} 55' 45''$, and $C=111^{\circ} 6' 15''$

217. Given $a=87^{\circ} 10' 15''$, $c=100^{\circ} 10' 15''$, and $B=61^{\circ} 31' 15''$; required b . *Ans.* $b=62^{\circ} 36' 45''$

218. Given $a=81^{\circ} 10'$, $A=81^{\circ} 24'$, and $b=62^{\circ} 36' 45''$; required B. *Ans.* $B=62^{\circ} 41'$

219. Given $a=49^{\circ} 10'$, $b=58^{\circ} 25'$, and $C=71^{\circ} 18' 30''$; required the angles A and B. *Ans.* $A=59^{\circ} 2'$, $B=74^{\circ} 54'$

220. Given $a=87^{\circ} 10' 15''$, $c=100^{\circ} 10' 15''$, and $B=61^{\circ} 31' 15''$; required the angles A and C.

Ans. $A=81^{\circ} 24'$, $C=102^{\circ} 59'$

221. Given $A=59^{\circ} 2'$, $B=74^{\circ} 54'$, and $c=56^{\circ} 42'$; required the sides a and b . *Ans.* $a=49^{\circ} 10'$, $b=58^{\circ} 25'$

222. Given $A=115^{\circ} 38' 45''$, $C=75^{\circ} 31' 30''$, and $b=108^{\circ} 4' 15''$; required the sides a and c .

Ans. $a=119^{\circ} 42' 15''$, $c=68^{\circ} 53' 45''$

IV.

223. In the right-angled spherical triangle ABC, given $B=60^{\circ} 32' 15''$, $a=94^{\circ} 57' 20''$, and $A=90^{\circ}$; required the other parts. *Ans.* $C=98^{\circ} 41' 45''$, $c=100^{\circ}$, $b=60^{\circ} 10'$

224. Given $a=77^{\circ} 52' 10''$, $b=74^{\circ} 19' 30''$, and $A=90^{\circ}$; to find the other parts.

Ans. $c=38^{\circ} 56' 24''$, $B=80^{\circ}$, $C=40^{\circ}$

225. In the quadrantal triangle ABC, given $a=90^\circ$, $B=74^\circ 36' 30''$, and $c=50^\circ 10'$; required the other parts.

Ans. $A=100^\circ$, $b=78^\circ 14' 25''$, $C=49^\circ 8' 15''$

226. Given $a=90^\circ$, $A=96^\circ 17' 52''$, $b=82^\circ 26'$; required the other parts. *Ans.* $B=80^\circ 10'$, $C=50^\circ 2'$, $c=50^\circ 27'$

V.

227. Required the product of $2.4 \times .0035 \times 1.08 \times .1$.

Ans. .0009072

228. Divide 9.5 by .36.

Ans. 26.389

229. Find the value of $\frac{8.4 \times .0769 \times .00683}{59.8 \times .0000146 \times .0039}$

Ans. 1295.71

230. Find the 3d power of .2321 and the 72d power of .96797.

Ans. .0125 and .096

231. Required the 200th root of .0063241. *Ans.* .975

232. Required the value of $\sqrt[7]{.6958825}$. *Ans.* .00563

233. Required the value of the following expression :

$$\sqrt{\frac{\sqrt[7]{.096} + (.096)^{72}}{2.5}}$$

Ans. .29904

234. In the plane triangle ABC, given $a=\frac{1}{8}$, $b=.2705$, and $c=.3375$; required the angle C. *Ans.* $C=111^\circ 47' 45''$

235. In the plane triangle ABC, given $a=116$, $b=172.5$, and $A=37^\circ 20'$; to find B. *Ans.* $B=64^\circ 24'$, or $115^\circ 36'$

236. In the plane triangle ABC, given $a=.512$, $b=.627$, and $C=42^\circ 53' 38''$; required the angles A and B.

Ans. $A=54^\circ 8' 11''$, $B=82^\circ 58' 11''$

237. Given $b=.2$, $c=.25$, and $A=22^\circ 20'$; required the third side a. *Ans.* $a=.1$

238. In the right-angled plane triangle ABC, given $a=177\frac{1}{2}$, $c=117.7$, and $B=90^\circ$; required the side b.

Ans. $b=213.21$

239. In the trapezium ABCD (fig. p. 54), given the side $AB=90$ yards, $BC=100$ yards, $CD=110$ yards, $DA=120$ yards, and the angle $DAB=116^\circ$; required the area.

Ans. Area=9768.7 yards.

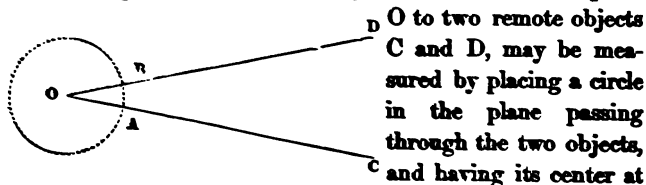
CHAPTER IV.

PROBLEMS IN ASTRONOMY, SURVEYING, AND NAVIGATION.

Application of the Rules and Formulae in Plane and Spherical Trigonometry.

(44). When a line joining any two points in space is accessible throughout its whole extent, it may in general be measured by the successive application of some line of a known length ; but when it is inaccessible, or cannot be directly measured, we may obtain its length by considering it the *side of a triangle*, if we already know, or can find by observation, a sufficient number of parts of that triangle to enable us to apply one or more of the preceding rules. It is thus that the mensuration of inaccessible lines of any length is found by means of that of accessible lines and angles.

The angle DOC , contained by lines drawn from a point O to two remote objects C and D , may be measured by placing a circle in the plane passing through the two objects, and having its center at the angular point O . The straight edge of a ruler being then placed on the circle so as to pass through its center, and, by means of sights placed over it, directed first to one object and then to the other, the arc AB of the circumference between the two positions of the ruler can be found ; this is the measure of the angle O .



(45). The principal instruments for measuring angles are the theodolite and sextant. A Theodolite is the most convenient instrument for measuring horizontal and vertical angles :

it is composed of two circles having their planes perpendicular to each other. When the instrument is used, one of the circles is placed in a horizontal plane, by means of levels ; on this circle horizontal angles are measured ; on the other are measured vertical angles, whether of elevation or depression (that is, whether the object is above or below the horizontal line). A Sextant is employed to measure angles contained in any plane whatever. It is more suited for observing angular distances of heavenly bodies than the theodolite ; but the latter is better adapted for Surveying than the former, since it determines the horizontal angles at once ; but those observed with the sextant must, when out of the plane of the horizon, be reduced to that plane by calculation, to suit them to the purposes of the survey.

We will not stop to give particular descriptions of the theodolite and sextant : their construction will be best learned by a careful study of the instruments themselves ; we shall therefore suppose the manner of adjusting and applying them to practice is known, and proceed to give a collection of problems in which these instruments have supplied for the most part the necessary data.

SECTION I.

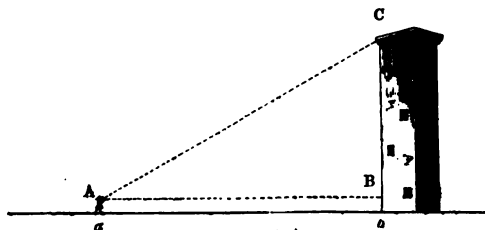
Problems in Surveying, &c.

The rules of Plane Trigonometry just given will enable us to solve many useful and interesting problems in Surveying and Navigation. A large collection of these will be found at the end of the book ; they have been selected to serve as an introduction to Navigation and Nautical Astronomy. We will give, however, in this place a few that most frequently occur in practice, with their solutions at length ; and as it is often very necessary that a naval student should know how to solve trigonometrical problems, not only by logarithms, but also *instrumentally*, that is, by constructing the figures with

the aid of mathematical instruments, we will make use of both methods when we come to solve some of the problems in Navigation.

1. Wishing to find the height of a tower, I observed the angle of elevation of its top above the level of my eye to be $32^{\circ} 14'$. I then measured the distance ab from the place of observation to the base of the tower, and found it to be 142 feet. Required the height of the tower.

Let CB represent the tower, and A the place of the observer: draw the horizontal line AB at a height above the ground equal to the height of the eye, and join AC . Then in



the right-angled triangle ABC are given $AB=142$ feet, the angle $CAB=32^{\circ} 14'$, and the angle $B=90$; to find CB , the height of the tower above the horizontal line AB .

By Rule, p. 50 $\frac{CB}{AB} = \tan. A$

$$\therefore CB = AB \tan. A.$$

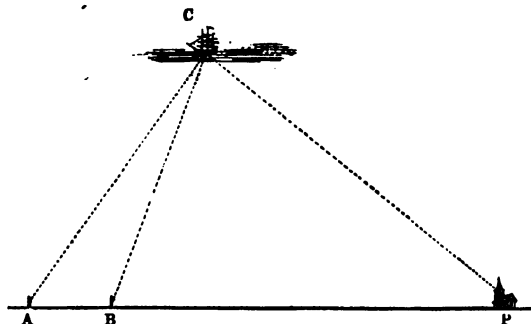
$AB=142$	$\therefore \log. CB = \log. AB + \log. \tan. A - 10$
$A=32^{\circ} 14'$	„ $AB \dots\dots 2.152288$
	„ $\tan. A \dots\dots 10.072690$
	„ $CB \dots\dots 2.224978$
	$\therefore CB=167.9$ feet.

To the value of CB must be added the height of the eye Aa : the result will be the height of the tower required.

2. Being on Southsea Common, and wishing to find my

distance from a ship at anchor at Spithead, I observed with a quadrant the angle between the ship and the steeple of St. Thomas's Church to be $72^{\circ} 42'$. I then walked 500 yards in a direct line towards the church, and again took the angle between the ship and the steeple, and found it to be $82^{\circ} 45'$. Required my distance from the ship at each observation.

Let C be the ship at anchor, A my first station, B the second, and P the church. Then the angle $CAP = 72^{\circ} 42'$,



$CBP = 82^{\circ} 45'$, and the line $AB = 500$ yards, are given to find AC and BC , my distances at each observation.

In the triangle ABC , the angle $ABC = 180^{\circ} - CBP = 97^{\circ} 15'$
 $\therefore$ angle $C = 180^{\circ} - (72^{\circ} 42' + 97^{\circ} 15') = 10^{\circ} 3'$.

(1.) To find BC .

By Rule, p. 41, $BC : AB :: \sin. A : \sin. C$

$AB = 500$	$\log. AB \dots\dots 2.698970$
$A = 72^{\circ} 42'$	$,, \sin. A \dots\dots 9.979895$
$C = 10^{\circ} 3'$	$ \underline{12.678865}$
	$,, \sin. C \dots\dots 9.241814$
	$,, BC \dots\dots 3.437051$
	$\therefore$ distance $BC = 2735$ yards.

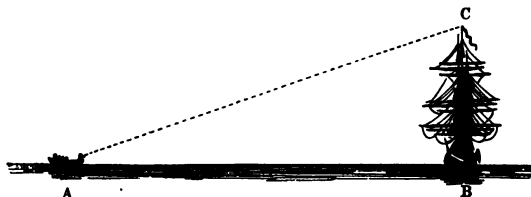
(2.) To find AC .

$AC : AB :: \sin. ABC : \sin. C$

AB=500	log. AB.....	2.698970
ABC=97° 15'	„ sin. ABC ..	9.996514
C=10 3		<u>12.695484</u>
	„ sin. C.....	9.241814
	„ AC	3.453670
	∴ distance AC=	2841 yards.

3. Being ordered to place a target at 500 yards from the ship, and knowing that the height of the truck above the water-line was 213 feet, it is required to find what angle this height will subtend on my sextant when I am at the required distance (before allowing for index correction).

Let BC represent the ship's mast, A the required place of



the target: then the angle BAC is the angle which must be read off on the sextant.

In the right-angled triangle ABC are given the side BC=213 feet, AB=500 yards, or 1500 feet, and B=90°; to calculate the angle A.

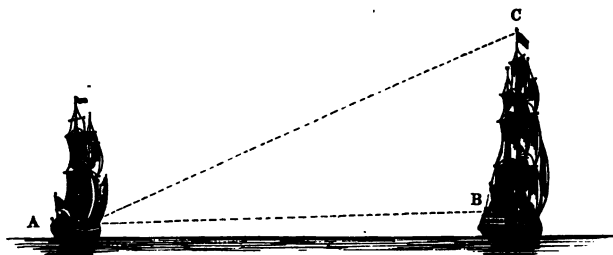
By Rule, p. 50, $\tan. A = \frac{BC}{AB}$

BC= 213	∴ log. tan. A-10=log. BC-log. AB
AB=1500	or log. tan. A=10+log. BC-log. AB
	log. BC+10.....12.328380
	„ AB 3.176091
	„ tan. A <u>9.152289</u>
	∴ A=8° 5'

4. Sailing in company with another ship, and being ordered to keep at the distance of 500 yards from her, and knowing that the height of her mast above the hammock nettings was

198 feet, it is required to find what angle on my sextant will indicate the proper distance.

Let A and B be the two ships, A the place of the observer, and CAB the angle subtended by the mast CB at A. Then



in the right-angled triangle CAB are given $CB=198$ feet, $AB=1500$ feet, and $B=90^\circ$; to find the angle CAB.

By Rule, p. 50, $\tan. A = \frac{CB}{AB}$

$CB = 198$

$\therefore \log. \tan. A - 10 = \log. CB - \log. AB$

$AB = 1500$

$\log. \tan. A = 10 + \log. CB - \log. AB$

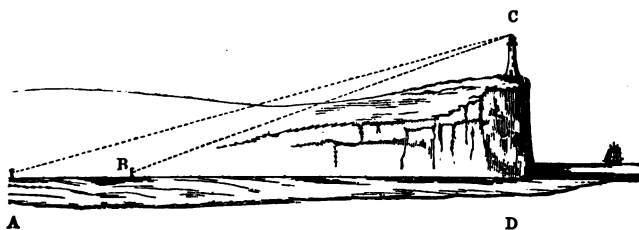
$\log. CB + 10 \dots\dots 12.296665$

„ $AB \dots\dots 3.176091$

„ $\tan. A \dots\dots 9.120574$

$\therefore A = 7^\circ 31' 15''$

5. Wishing to determine the height of a lighthouse C on



the summit of a cliff on the sea-shore, I observed the angle of elevation CAD of its top above the level sand to be $26^\circ 40'$:

then, measuring on the sand in a direct line towards it a line $AB=200$ yards, I again observed the angle of elevation of its top, and found it to be $33^\circ 30'$. Required the height CD of the lighthouse above the shore.

(1.) In the triangle CAB are given $AB=200$, angle $CAB=26^\circ 40'$, angle $ABC=180^\circ-33^\circ 30'=146^\circ 30'$, and therefore the remaining angle $ACB=6^\circ 50'$; to find the side CB .

By Rule, p. 41, $CB : AB :: \sin. A : \sin. ACB$.

$AB=200$	$\log. AB \dots\dots\dots$	2.301030
$A=26^\circ 40'$	$,, \sin. A \dots\dots\dots$	9.652052
$ACB=6^\circ 50'$		11.953082
	$,, \sin. ACB \dots\dots\dots$	9.075480
	$,, CB \dots\dots\dots$	2.877602
		$\therefore CB=754.4 \text{ yards.}$

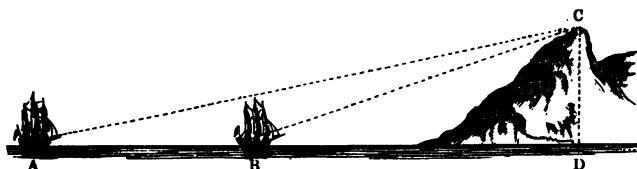
(2.) In the right-angled triangle CBD are given $\log. CB=2.877602$, angle $CBD=33^\circ 30'$, and $D=90^\circ$; to find CD , the height of the lighthouse.

By Rule, p. 50, $\frac{CD}{CB} = \sin. CBD$

	$\therefore CD = CB \sin. CBD$
	$\therefore \log. CD = \log. CB + \log. \sin. CBD - 10$
	$\log. CB \dots\dots\dots$
	2.877602
	$,, \sin. CBD \dots\dots\dots$
	9.741889
	12.619491
	10
	$,, CD \dots\dots\dots$
	2.619491
	$\therefore CD=416.3 \text{ yards.}$

6. Standing in for the land, I observed the summit of a lofty mountain near the shore. I took the angle of elevation of the peak, and found it to be $12^\circ 25'$; after having run $3\frac{1}{2}$ miles directly towards it, I again took the angle of elevation, which was $30^\circ 13'$. Required the height of the mountain, and its distance from the second station.

Let CD represent the mountain, A the ship's place at the first observation, B the place at the second observation. Then, in the triangles CAB and CBD we have given the side



$AB=3.5$ miles, the angle $CAB=12^\circ 25'$, and the angle $CBD=30^\circ 13'$; to find the height of the mountain CD , and the distance DB from the station B .

(1.) To find BC .

In the triangle ABC , $BC:AB::\sin. BAC:\sin. ACB$.

$AB=3.5$	$\log. AB \dots\dots\dots 0.544068$
$BAC=12^\circ 25'$	$,, \sin. BAC \dots\dots 9.332478$
$ACB=CBD-BAC$	$\underline{9.876546}$
$\therefore CBD=30^\circ 13'$	$,, \sin. ACB \dots\dots 9.485289$
$BAC=\underline{12 \quad 25}$	$,, BC \dots\dots\dots 0.391257$
$\therefore ACB=\underline{17 \quad 48}$	$\therefore BC=2.46 \text{ miles.}$

(2.) To find CD .

In the right-angled triangle CBD , $\frac{CD}{BC}=\sin. CBD$

$$\therefore CD=BC \cdot \sin. CBD$$

$BC=2.46$	$\log. BC \dots\dots\dots 0.391257$
$CBD=30^\circ 13'$	$,, \sin. CBD \dots\dots 9.701802$
	$,, CD \dots\dots\dots 0.093059$
	$\therefore CD=1.24 \text{ miles.}$

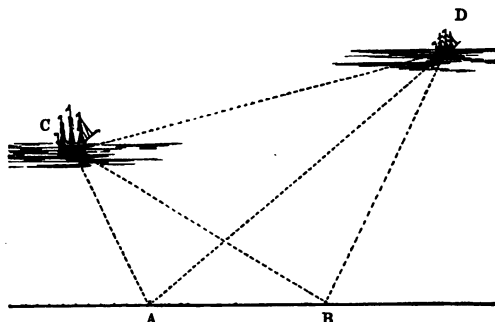
(3.) To find DB .

In the right-angled triangle CBD , $\frac{BD}{BC}=\cos. CBD$

$$\therefore BD=BC \cdot \cos. CBD$$

BC=2.46	.	log. BC.....0.391257
CBD=30° 13'		„ cos. CBD ..9.936578
		„ BD.....0.327835
		∴ BD=2.13 miles.

7. Wishing to know the distance between two ships at anchor at C and D, I measured on the shore a base line AB=735 feet, and with a sextant observed the following angles.



At A, one end of the base, CAD=63° 30', and DAB=35° 10'; at B, the other end of the base, DBC=80° 16', and CBA=28° 20'. Required CD, the distance between the two ships.

- (1.) In triangle ACB, find AC.
- (2.) In triangle ABD, find AD.
- (3.) In triangle ACD, find CD.

(1.) To find AC.

In triangle ACB are given AB=735, CBA=28° 20', CAB=63° 30' + 35° 10' = 98° 40', and therefore the remaining angle ACB=53°.

CBA= 28° 20'	AC : AB :: sin. CBA : sin. ACB
CAB= 98 40	log. AB 2.866287
127 0	„ sin. CBA .. 9.676328
180	12.542615
ACB= 53	„ sin. ACB .. 9.902349
AB=735	„ AC 2.640266
	∴ AC=436.8

(2.) To find AD.

In the triangle ABD are given $AB=735$, $DAB=35^\circ 10'$, $ABD=28^\circ 20' + 80^\circ 16' = 108^\circ 36'$, and therefore the remaining angle $ADB=36^\circ 14'$.

DAB=	$35^\circ 10'$	AD : AB :: sin. ABD : sin. ADB
ABD=	$108^\circ 36'$	log. AB 2.866287
	$143^\circ 46'$	„ sin. ABD .. 9.976702
	180°	12.842989
ADB=	$36^\circ 14'$	„ sin. ADB .. 9.771643
AB=	735	„ AD 3.071346
		$\therefore AD=1178.5$

(3.) To find CD.

In the triangle ACD are given $AC=436.8$, $AD=1178.5$, and the included angle $CAD=63^\circ 30'$.

By Rule IV. p. 49.

AD	1178.5	CAD	$63^\circ 30'$
AC	436.8	$\therefore \frac{1}{2}$ CAD	31 45
AD+AC ..	1615.3		
AD-AC ..	741.7		

[1.]

log. (AD+AC) ..	3.208253
„ (AD-AC) ..	2.870226
	0.338027
„ tan. $\frac{1}{2}$ CAD ..	9.791563
„ tan. arc	10.129590

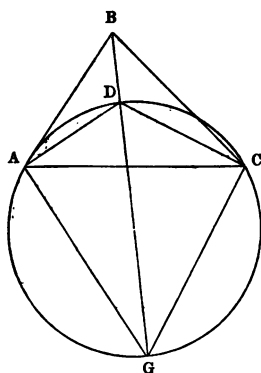
[2.]

log. (AD+AC) ..	3.208253
„ sin. $\frac{1}{2}$ CAD ..	9.721162
	12.929415
„ sin. arc	9.904757
„ CD	3.024658
$\therefore$ CD =	1058.5

As the two following problems are of great use in Marine Surveying, we will solve them by logarithms, and also by a geometrical construction. In problem 98 of the volume of *Astronomical Problems*, an analytical solution of the same problems is given.

8. Wishing to determine the position of a sunken rock at the

entrance of a bay, and the water being smooth, I anchored a boat



upon it, and measured with a sextant the angles which three objects, A, B, and C, on the shore subtended at the boat. They were as follows: the angle between A and the object B to the right was $26^{\circ} 27'$, and the angle between B and the object C to the right was $34^{\circ} 12'$. On my chart of the bay I carefully measured with compasses the distances between the three objects, and found $AB=5$ miles, $BC=6$ miles, and $AC=7$ miles. Required

the distance of the rock from each object.

(1.) *By Construction.*

By means of a scale of equal parts make the triangle ABC, having the side $AB=5$, $BC=6$, and $AC=7$. At the point C, on the side of AC farthest from the boat, make the angle $ACD=26^{\circ} 27'$, the angle observed between the other two objects A and B; and at the point A, on the farthest side also from the boat, make the angle $CAD=34^{\circ} 12'$, the angle between the other two objects B and C: produce the sides AD and CD till they meet in D. Then describe a circle to pass through the three points A, D, and C; and the position of the rock will be somewhere in the circumference of that circle. To find the place of the rock, join BD, and produce it to the circumference in G; then G will be the station sought, or the position of the rock.

For, the angles in the same segment of a circle being equal (*Euclid*, b. iii.), therefore $AGB=ACD=26^{\circ} 27'$, and $CGB=CAD=34^{\circ} 12'$; and these were the angles observed at the boat. Hence G must be the position of the boat; and GA, GB, and GC measure respectively the distance of each point from the rock.

(2.) *By Trigonometry.*

Assume any point G to be the position of the boat, and let A, B, and C be the objects on shore. Describe a circle passing through the three points A, G, C. Join BG, GA, and GC. Then GA, GB, and GC are the distances required. Draw AD, CD to the point of intersection D. Then, by Geometry, since the angles in the same segment of a circle are equal, $\therefore CAD = CGB = 34^\circ 12'$, and $ACD = AGB = 26^\circ 27'$.

[1.] Find AD, having given in the triangle ADC the side $AC=7$, the angle $ACD=26^\circ 27'$, and $ADC=180-(34^\circ 12' + 26^\circ 27')=119^\circ 21'$.

[2.] Find angle BAC, having given the three sides of the triangle ABC.

[3.] Find angle ABD, having given AB, AD, and angle BAD ($=BAC-CAD$).

[4.] Find GA, GB, and angle BAG, having given in the triangle ABG the side AB and the angles AGB and ABG.

[5.] Find GC, having given in the triangle AGC the side AC, the angle AGC, and the angle $CAG=(BAG-BAC)$.

Calculation.

[1.] To find AD:		[2.] To find angle BAC.	
AC : AD :: sin. ADC : sin. DCA		7	9.154902
AC=7	0.845098	AB=5	5 9.301030
ADC=119° 21'	9.648766	BC=6	2 0.602060
DCA= 26 27	10.493864	AC=7	6 0.301030
	9.940338		8 9.359022
	0.553526	4	BAC=57° 7' 15"
$\therefore AD=3.577$		4	CAD=34 12 0
		2.	BAD=22 55 15

[3.] To find angle ABD.

$$AB+AD:AB-AD::\tan.\frac{1}{2}(ADB+ABD):\tan.\frac{1}{2}(ADB-ABD)$$

AB=5	0.153205
AD=3.577	10.692995
<u>8.577</u>	<u>10.846200</u>
1.423	0.933335
BAD= 22° 55' 15"	<u>9.912865</u>
180	39° 17' 15"
<u>157 4 45</u>	<u>78 32 22</u>
78 32 22	ADB=117 49 37
	ABD= 39 15 7

[4.] To find GA, GB, and the angle BAG.

AB=5	AB : GA :: sin. AGB : sin. ABG	
AGB= 26° 27'	AB : GB :: sin. AGB : sin. BAG	
ABG= 39 15	0.698970	0.698970
<u>65 42</u>	<u>9.801201</u>	<u>9.959711</u>
180	10.500171	10.658681
BAG=114 18	9.648766	9.648766
	<u>0.851405</u>	<u>1.009915</u>
	GA=7.103	GB=10.23

[5.] To find GC.

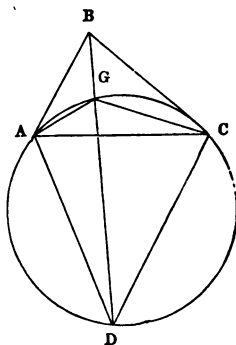
	AC : GC :: sin. AGC : sin. CAG	
AC=7	0.845098	
AGC= 60° 39'	9.924491	
BAG=114 18	<u>10.769589</u>	
BAC= 57 7	9.940338	
∴ CAG= 57 11	<u>0.829251</u>	
	∴ GC=6.75	

9. Sailing in a deep and unknown bay, I suddenly found the soundings decrease ; and suspecting I was on a coral reef, I hauled off, having anchored a boat on it, from which the following angles were taken between three remarkable objects, A, B, and C, that appeared on the distant shores, namely : between A and B, a high-pointed rock to the right of A, the angle was 116° 40' ; between B and C, a bluff summit to the

right of B, the angle was $112^{\circ} 30'$. I obtained afterwards the distances between the three points A, B, and C, by measuring on the shore convenient base lines, and taking angles, as pointed out in Example, p. 80. The distance from A to B was 5.75 miles, from B to C 7.5 miles, and from A to C 8.25 miles. It is required to find the position of the reef.

(1.) *By Construction.*

Construct the triangle ABC from the scale of equal parts, by taking $AB=5.75$, $BC=7.5$, and $AC=8.25$. At the point C, in the straight line AC, make $ACD=63^{\circ} 20'$, the supplement of the angle subtended by the other two points A and B. Again, at the point A, in the straight line AC, make $CAD=67^{\circ} 30'$, the supplement of the angle subtended by the other two points B and C: produce the lines AD and CD to meet in the point D. About the triangle ADC describe a circle; then the place of the reef will be somewhere in the circumference of this circle. To find it, join BD; and the point of intersection G is the position of the rock required.



For since the angles in the same segment of a circle are equal (*Euclid*, b. iii.), therefore $AGD=ACD=63^{\circ} 20'$; therefore the angle $AGB=116^{\circ} 40'$. Again, $CGD=CAD=67^{\circ} 30'$; therefore the angle $BGC=112^{\circ} 30'$. And these were the angles observed at the boat; therefore G must be the place of observation.

(2.) *By Trigonometry.*

Assume any point G as the position of the reef, and let A, B, and C be the objects on shore. Describe a circle passing through the three points A, G, and C. Join BG, and produce it to meet the circle in D. Join GA and GC. Then GA, GB, and GC are the distances required. Join AD

and CD. Then, by Geometry, since the angles in the same segment are equal, $\therefore$ angle $DAC = DGC = 180 - BGC = 180^\circ - 112^\circ 30' = 67^\circ 30'$, and angle $ACD = AGD = 180 - AGB = 180^\circ - 116^\circ 40' = 63^\circ 20'$.

[1.] Find AD, having given in the triangle ADC, $AC = 8.25$, angle $ACD = 63^\circ 20'$, and angle $ADC = 180^\circ - (67^\circ 30' + 63^\circ 20') = 49^\circ 10'$.

[2.] Find the angle BAC, having given the three sides of the triangle ABC.

[3.] Find the angle ABD, having given AB, AD, and the angle $BAD = (BAC + CAD)$.

[4.] Find GA, GB, and the angle BAG, having given in the triangle ABG the side AB and the angles AGB and ABG.

[5.] Find GC, having given in the triangle AGC the side AC, the angle AGC, and the angle $CAG = BAC - BAG$.

Calculation.

[1.] To find AD.

$$AC : AD :: \sin. ADC : \sin. DCA$$

AC = 8.25		0.916454
ADC = $49^\circ 10'$		9.951159
DCA = $63^\circ 20'$		10.867613
		9.878875
		0.988738
	$\therefore AD =$	9.744

[2.] To find the angle BAC.

AB = 5.75	8.25	9.083546
AC = 8.25	5.75	9.240332
BC = 7.5	2.50	0.698970
	7.5	0.397940
	10.0	9.420788
	5.0	BAC = $61^\circ 46' 15''$
	5.0	CAD = $67^\circ 30' 0''$
	2.5	BAD = $129^\circ 16' 15''$

[3.] To find the angle ABD.

$$AD + AB : AD - AB :: \tan. \frac{1}{2}(ABD + ADB) : \tan. \frac{1}{2}(ABD - ADB)$$

AD = 9.744	0.601408
AB = 5.750	9.675890
<u>15.494</u>	<u>10.277298</u>
3.994	1.190164
180° 0' 0"	9.087134
BAD = 129 16 15	6° 58'
<u>50 43 45</u>	<u>25 22</u>
25 21 52	∴ ABD = 32 20

[4.] To find GA, GB, and the angle BAG.

AB = 5.75	AB : GA :: sin. AGB : sin. ABG	
AGB = 116° 40'	AB : GB :: sin. AGB : sin. BAG	
ABG = 32 20	0.759668	0.759668
<u>149 0</u>	<u>9.728227</u>	<u>9.711839</u>
180	10.487895	10.471507
BAG = 31 0	<u>9.951159</u>	<u>9.951159</u>
	0.536736	0.520348
	∴ GA = 3.44	∴ GB = 3.31

[5.] To find GC.

AC = 8.25	AC : GC :: sin. AGC : sin. CAG	
AGB = 116° 40'	BAC = 61° 46'	0.916454
BGC = 112 30	BAG = 31 0	9.708882
<u>229 10</u>	∴ CAG = 30 46	<u>10.625336</u>
360		9.878875
∴ AGC = 130 50		<u>0.746461</u>
		∴ GC = 5.58

Problems in Navigation.

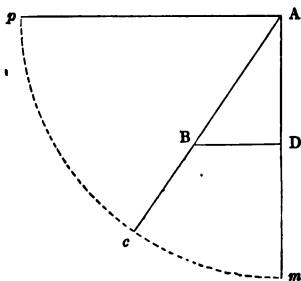
It will be proved in *Navigation*, Part II., that the course, distance, true difference of latitude, and departure between any two places on the surface of the earth, may be correctly represented by the sides and angles of a right-angled plane triangle. Let us suppose A and B to be the two places, and

courses, the several points of the compass must be thoroughly known. An expeditious method of forming the compass, and of learning it, is given in *Navigation*, Part I. p. 11.

10. A ship from latitude $47^{\circ} 30' N.$ has sailed S.W.b.S. 98 miles: find by construction, and by calculation, the latitude she is in, and the departure she has made.

(1.) *By Construction.*

Let A represent the point the ship departed from, AD the meridian, and Ap, drawn at right angles to it, the parallel of latitude of the ship. At the point A, with the chord of 60° , describe the quadrant mp, and cut off mc=S.W.b.S. or $33^{\circ} 45' =$ the course; and through c draw a line AB. From a scale of equal parts take AB=98 miles, the distance; and through B draw BD parallel to Ap, meeting AD in D. Then B is the place the ship has arrived at, AD is the difference of latitude, and BD is the departure. If AD and BD are measured by the same scale of equal parts, it will be found that the difference of latitude AD is about 81 miles, and the departure BD about 54 miles. The figure may be more easily laid off by means of a protractor (see *Practical Geometry*).



(2.) *By Trigonometry.*

In the right-angled triangle ABD are given the course $DAB=33^{\circ} 45'$, and distance $AB=98$ miles; to find the difference of latitude AD, and departure BD.

$$\text{By Rule, p. 50, } \frac{AD}{AB} = \cos. DAB$$

$$\begin{aligned} AB &= 98 & \therefore AD &= AB \cos. DAB \\ DAB &= 33^{\circ} 45' \end{aligned}$$